Structural Mechanics II

Part I: Structural Stability and Structural Dynamics

Vincent Denoël

GCIV2037-1

Academic Year 2017-2018

Last update : February 7, 2018

General Information

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Where? When?

The course is organized on Thursday, 8:30-12:30 (second semester)

Contact me?

Office room: B52/3, +1/422
Phone: 04/366.29.30
Mail: v.denoel@ulg.ac.be

Objectives of the course and pedagogical plan? System of examination?

5 ECTS includes classes and homeworks (more details in “Engagements pédagogiques”)
Oral and written examination

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Organization

References:
Dynamics of Structures, Clough and Penzien
Structural Stability, Timoshenko
Analyse des structures II, V. Denoël (available lecture notes)

Course Outline:

- Structural Stability
- Numerical Integrators (solution of IVP)
- Deterministic Dynamics
  - Discrete Models: single- and multiple-degree-of-freedom systems
  - Continuous Models
- Introduction to Seismic Analysis

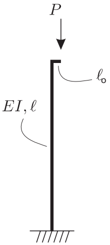
Tentative Agenda

Feb 8th, 2017	Structural stability
Feb 15th, 2017	Structural stability & Intro dynamics
Feb 22rd, 2017	Plates (Prof. Rigo)
Mar 1st, 2017	Plates (Prof. Rigo)
Mar 8th, 2017	Plates (Prof. Rigo)
Mar 15th, 2017	Eq. of motion, Numerical Integrators
Mar 22rd, 2017	Numerical Integrators
Mar 29th, 2017	SDOF systems: Free response/Harmonic loading
Apr 5th, 2017	Respiration
Apr 12th, 2017	Easter **Test - Stability**
Apr 19th, 2017	SDOF systems: Impulsive loading
Apr 26th, 2017	MDOF systems
May 3th, 2017	MDOF systems
May 10th, 2017	Ascension
May 17th, 2017	Continuous systems, Intro to Earthquake Engineering

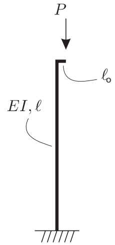
Section I: Introduction

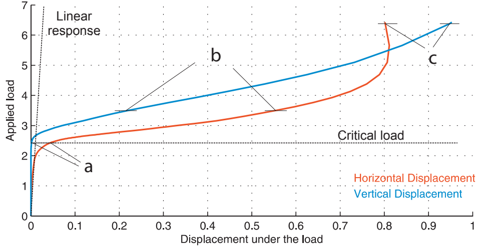
Learning Outcomes:

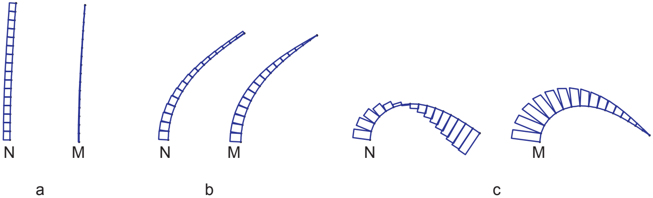
- understand the balance of forces in structural dynamics
- overview of possible applications of the theories developed in this course
- gain practice with the lumped modeling of real structures



$EI = 1,\ \ell = 1,\ \ell_o = 2 \cdot 10^{-3},\ EA = 10^6$

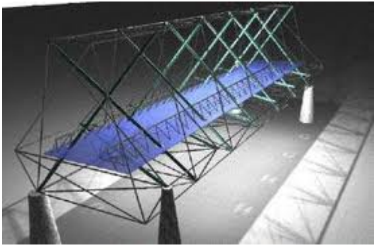


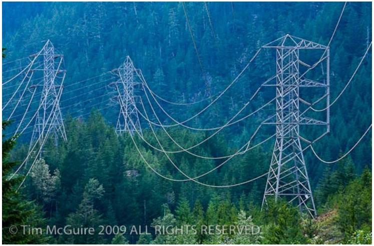


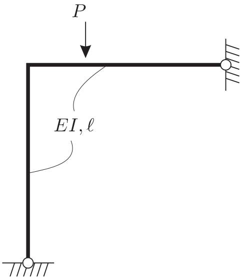


$$EI = 1,\ \ell = 1,\ \ell_o = 2\cdot 10^{-3},\ EA = 10^6$$

Nonlinear Structural Analysis III







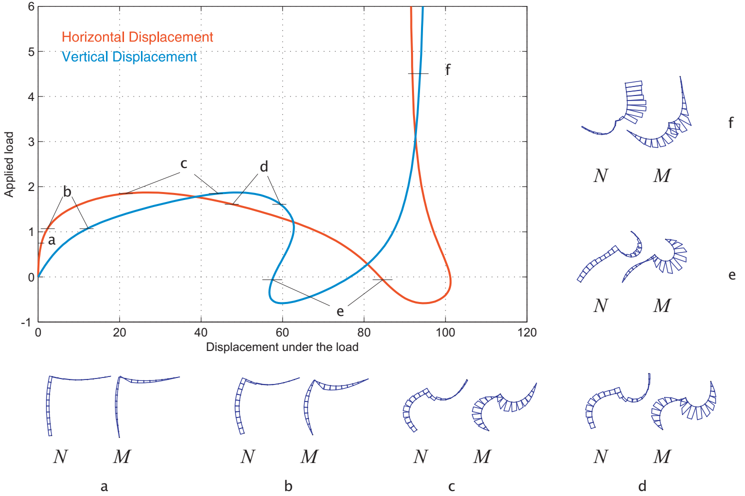
Lee's frame

Displacement under applied load ?

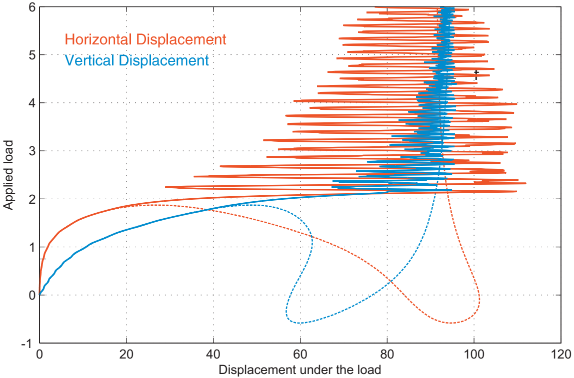
[axial force is column is less than P, but
(i) it is also bent,

(ii) the beams stabilizes the column (rotational spring)]

Nonlinear Structural Analysis V

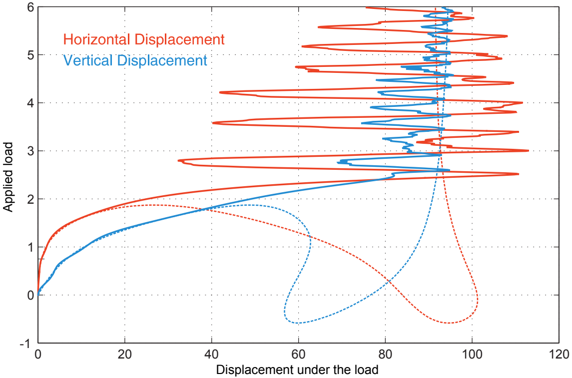


Nonlinear Structural Analysis VI



$\rho = 1$

Nonlinear Structural Analysis VII



$\rho = 10$

Real-life Examples of Stability Issues I

— Show pictures and videos —

Objectives I

Nonlinear Analysis of Structures = determination of internal stresses, displacements, strains, etc.

Nonlinear → Equilibrium in a deformed configuration

Classification:

- large displacement (necessary - to be discussed when setting up equations)
- large or moderate rotations

Real-life Examples of dynamic loadings I

A few videos showing examples of application of structural dynamics in civil engineering

- Cable-stayed bridge Dubrovnik. `https://youtu.be/SsfQN1ilcGU`
- Galloping Basin Electric transmission lines due to wind `https://youtu.be/GEGbYRii1d4`
- Démolition Des Piliers Auxiliaires D'un Pont `https://youtu.be/SBdTwcbp69M`
- Amazing Controlled Demolition Of A skyscraper `https://youtu.be/e2E_m7l2Rww`
- Reach racking collapse `https://youtu.be/ZXvWARWM-OE`
- Wolfsburg Stadium roof during storm `https://youtu.be/69nR1QXQM6Q`
- Le stade de Francfort qui tremble ! `https://youtu.be/ITeOOHdN-zI`

Real-life Examples of dynamic loadings II

Dynamics in Civil Engineering

- Seismic loading
- Wind buffeting loading
- Aeroelastic and hydro-elastic phenomena (vortex shedding, galloping, rain-wind, etc)
- Footbridges & stadia: human-induced vibrations (comfort assessment)
- Impacts and explosions
- Wave loading, marine flows, breaking waves
- Traffic-induced vibrations in bridges (bridge-vehicle interactions, comfort)
- Vibrations resulting from turning and vibrating machines
- Vibrations resulting from sports activity (jump, dance, etc)
- Cable vibrations (anchor motion, wind action)

Objectives I

Dynamic Analysis of Structures = determination of internal stresses, displacements, strains, etc.

Dynamic $\rightarrow$ Time Evolution

(to be precised in the equations...)

Classification:

- Stationary .vs. Non-stationary Loadings
- Stationary .vs. Non-stationary Response
- Fast Dynamics .vs. Quasi-static Response (slow loads)
- Linear .vs. Nonlinear structural behavior (ex. cable dynamics)
- Deterministic .vs. Stochastic Dynamics

Objectives II

In a structural design

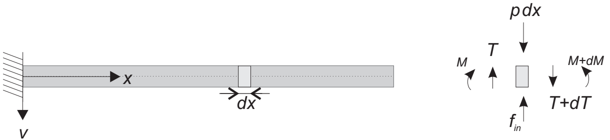
- use extreme values (in time series)
    - of stresses or internal forces (ULS)
    - of displacement/velocity/acceleration (SLS)
- choose a clever design
    - ductility
    - robustness, redundancy, alternative load paths

Warning: Although we study the structural analysis in this course, a clever design is of course always welcome. When uncertainties are coming into play (on the load level for instance), a clever design is usual much more interesting than a theoretically satisfied design.

The Governing Equations I

What do we solve ?

Example: Transverse vibrations of a Bernoulli beam



$$\mu \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v(x,t)}{\partial x^2}\right) - N \frac{\partial^2 v(x,t)}{\partial x^2} = p(x,t) \qquad (1)$$

with μ the lineic mass (mass per unit length) and EI the bending stiffness.

$$T = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI \frac{\partial^2 v}{\partial x^2}\right) \quad \text{and} \quad f_i = \mu\, dx \frac{\partial^2 v}{\partial t^2} \qquad (2)$$

[Beam equation, with inertial forces and geometric nonlinear forces]
[Same method applies for truss bars, plates, shells, volumes]

The Governing Equations II

Particular cases

- Nonlinear static analysis ($\partial \rightarrow d$) and constant EI

$$EIv'''' - Nv'' = p$$

- Linear static analysis

$$EIv'''' = p$$

- Linear dynamic analysis and constant EI

$$\mu \partial_t^2 v + EI \partial_x^4 v = p(x,t)$$

There are three ways to establish the governing equations:

- Global equilibrium
- Local equilibrium
- Energy considerations

The Governing Equations III

The governing equation of a continuous structure is a (partial) differential equation
The governing equation of a discrete structure is an algebraic (static) or differential equation (dynamic)

Examples: 1-DOF structure, 2-DOF structure, the Euler column (continuous structure)

The Governing Equations IV

In static structural analysis : How to get rid of the ODE ?

- use the displacement method (transforms into a set of algebraic equations)
- use the finite element method (same, more general)

In dynamic structural analysis : How to get rid of the PDE ?

Lumped modeling:

1. assume the response takes place in an **appropriate** shape $v(x,t) = \phi(x)\, q(t)$
   ($\phi(x)$ satisfies boundary conditions)
2. project the response in the assumed shape

$$m^{\star}\ddot{q}(t) + k^{\star} q(t)\, dx = \int_{\Omega} \phi(x)\, p(x,t)\, dx \tag{3}$$

with $m^{\star} = \int_{\Omega} \mu \phi^2(x)\, dx$ and $k^{\star} = \int_{\Omega} \phi(x) \frac{d^2}{dx^2}\left(EI \frac{d^2 v(x,t)}{dx^2}\right) dx$.

The Governing Equations V

[nb: Another way to cope with the difficulty of the PDE is to recourse to eigen functions - See section related to vibrations of continuous systems]

[nb: Another alternative is the finite element method (transformation of an ODE into a set of algebraic equations]

Section II: Structural Stability

Learning Outcomes:

- understand difference between bifurcation and divergence, critical and non-critical
- estimate critical load multipliers
- estimate second order displacements and internal forces

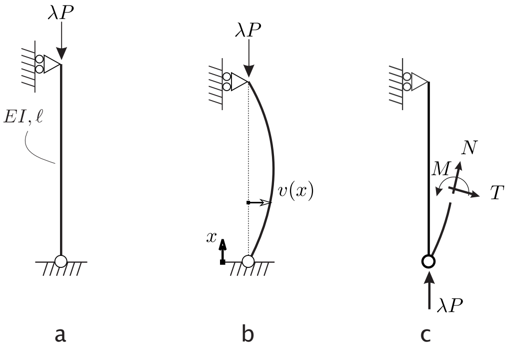
Méca II, 2017-2018

V. Denoël

Contents

A simple example to start with... I

Buckling of the Euler column



Governing equation (for example, from global equilibrium):

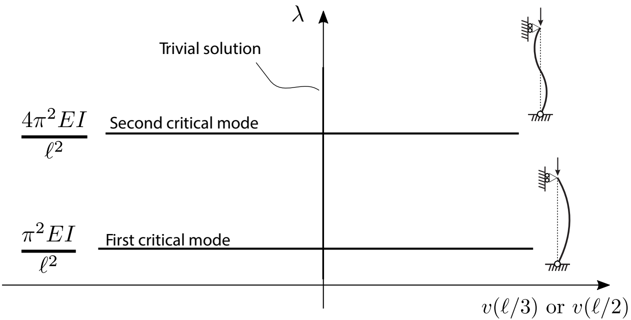
$$EI\, v'' + \lambda P\, v = 0$$

A SIMPLE EXAMPLE TO START WITH... II

Solution of the governing differential equation ($EI\,v'' + \lambda P\,v = 0$)

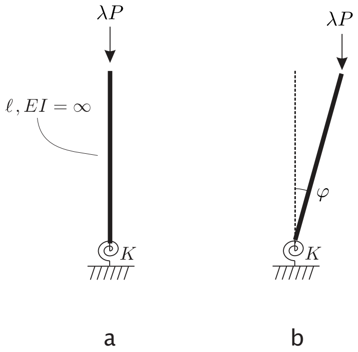
$$v = v_0 \sin\frac{k\pi x}{\ell} \qquad \text{and} \qquad \lambda P = k^2 \frac{\pi^2 EI}{\ell^2}$$

(or $v = 0$). What does *critical* mean ? There is a trivial solution $v = 0$ (because the equation has no righthand forcing term - it is homogenous)



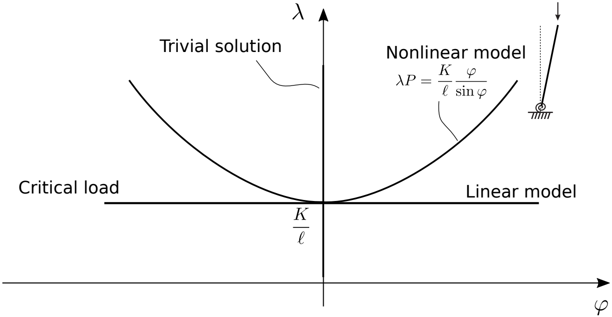
ANOTHER SIMPLE EXAMPLE... I

Determine the governing equation for this structure (use global equilibrium)



Investigate the nonlinear force-displacement response

Another simple example... II



Contents

Writing Equilibrium Equations I

Three different ways to obtain the governing equations

- Global equilibrium

$$EI\,v'' + \lambda P\,v = 0$$

- Local equilibrium

$$dM = Tdx \quad ; \quad dT = -Nd\theta \quad ; \quad dN = Td\theta$$

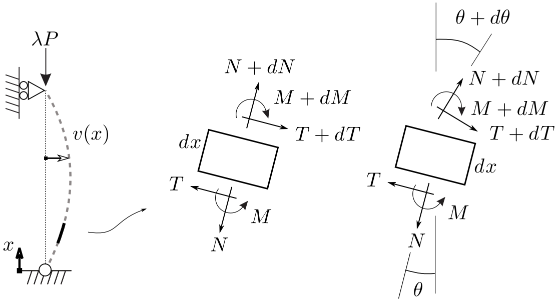
- Energy considerations (or Virtual works principle)

$$H[v(x); EI, \lambda P] := U + V = \int_0^{\ell} \left(EI\frac{v''^2}{2} - \lambda P\frac{v'^2}{2} \right) dx$$

All three methods provide the same governing equation

Writing Equilibrium Equations II

Local equilibrium



$$dM = Tdx \quad ; \quad dT = -Nd\theta \quad ; \quad dN = Td\theta$$

Discuss large **rotations** .vs. **large displacements** formulations

Writing Equilibrium Equations III

Energy considerations
The **total potential energy** is

$$H[v(x);\mathbf{p}] := U + V.$$

with U the potential of external forces (-work) and V the work done by internal forces. The equilibrium configurations of a structure $v_\star(x)$ are the stationary points of the total potential energy that satisfy the boundary conditions:

$$\frac{\partial H[v_\star + \varepsilon\eta]}{\partial\varepsilon} = 0$$

Example :

$$H[v(x);EI,\lambda P] = \int_0^\ell \left(EI\frac{v''^2}{2} - \lambda P \frac{v'^2}{2}\right) dx \quad \rightarrow \quad EI\, v_\star'''' + \lambda P\, v_\star'' = 0$$

Exercise. Do the same for the rigid column with rotational spring at its bottom support.

Writing Equilibrium Equations IV

Summary of important contributions to the total potential energy

- Truss bar / spring: $U = \frac{1}{2}\int_0^\ell EA\, u'^2 dx$, $U = \frac{1}{2}\frac{EA}{\ell}\Delta^2 = \frac{1}{2}k\Delta^2$
- Rotational spring: $U = \frac{1}{2}K\theta^2$
- Bernoulli beam : $U = \frac{1}{2}\int_0^\ell EI\, v''^2 dx$
- External Axial load : $V = \frac{1}{2}\lambda P \int_0^\ell v'^2 dx$

On the Derivation of the Governing Equation(s): Summary I

Two options to derive the governing equations

Local or global equilibrium
→ the governing equation translates the equilibrium of the body
→ Two possibilities:

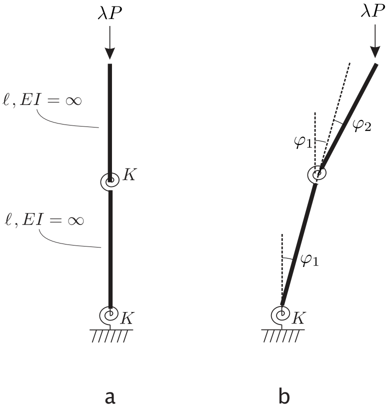
- ODE in case of continuous systems (infinite number of dofs) or
- Algebraic equation(s) in case of discrete systems

Conservation of energy
→ the governing equation translates the conservation of energy
→ the equilibrium equation may be recovered (but set it up right away if this is the objective)
→ deformed configuration(s) corresponding to equilibrium minimize(s) the total potential energy

- Investigate the nonlinear force-displacement response

Exercise

Determine the governing equations for this structure



- This structure has 2 degrees-of-freedom → set of 2 algebraic equations
- Compare global equilibrium and energy formulation

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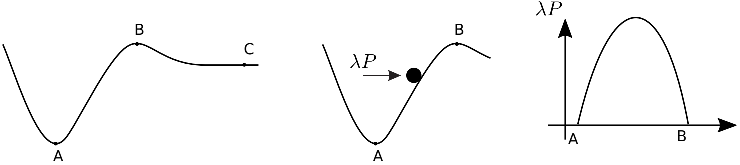
Stability of an Equilibrium Configuration I

The different types of stability



A: Stable - B: Unstable - C: Neutral

- Remember equilibrium means stationary point
- Transition from stable to unstable is associated with zero curvature

Stability of an Equilibrium Configuration II

An equilibrium configuration is said to be **stable** if the effects of an infinitely small perturbation tend to decrease asymptotically in time (nb: dynamics!?)

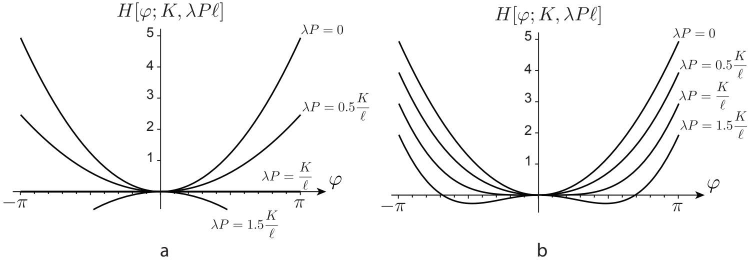
An equilibrium configuration is said to be **neutral** if the effects of an infinitely small perturbation remain indefinitely without growing nor decreasing (on "average")

Example. Let's perturb a bit the Euler column problem

$$EI\, v'' + \lambda P\, v = \varepsilon,$$

compute $v(x;\varepsilon)$, then look at the solution as $\varepsilon \to 0$.

Critical (Neutral) Stability Conditions



The Rayleigh Method I

- Discuss minimum of critical load multiplier

The Ritz-Galerkin Method I

- Replace a space-continuous system by a discrete algebraic model and compute eigen values

Section II: Numerical Integrators

Learning Outcomes:

- understanding of numerical techniques for initial value problems
- being able to implement some algorithms
- explain the limits of stability of integration schemes
- classification of numerical integrators into explicit or implicit families

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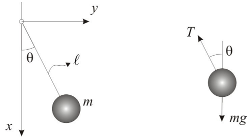
The Equation Of Motion I



kinetic energy : $T = \frac{1}{2} m r^2 \dot{\theta}^2$
potential energy: $-V = mgr\,(1 - \cos\theta)$

Conservation of energy

$$\frac{1}{2} m r^2 \dot{\theta}^2 + mgr\,(1 - \cos\theta) = \text{constant}$$

... differentiation with respect to t, then divide by $\dot{\theta}$

$$m r^2 \ddot{\theta} + mgr \sin\theta = 0 \qquad \text{with } \theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$$

Equation of motion: 2nd order differential equation (with 2 initial conditions)

The Equation Of Motion II

General Case

There are several ways to establish the equation of motion:

- Newton's law (second law, system of material points): $\sum \mathbf{f}_i = m\mathbf{a}$
- d'Alembert principle: $\sum \mathbf{f}_i - m\mathbf{a} = \mathbf{0}$
- principle of virtual works: $\delta W_I + \delta W_E = 0$
- Hamilton principle: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$ with $L = T - V$

In any case, one ends up with the same 2nd order differential equation (in displ., position, rotation)

In practice, a structure is a continuous medium. Two options:

- Model it as a continuous system (with one or several abscissa in space) → analytical developments
- Discretize the structural model with a finite number of degrees-of-freedom (e.g. finite element method).

The Equation Of Motion III

Discrete Version

Static Analysis (Finite Element Method):

$$\mathbf{K}\,\mathbf{x} = \mathbf{p} \tag{4}$$

Dynamic Analysis:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t) \tag{5}$$

- **M**: mass → inertia
- **C**: viscosity → damping
- **K**: stiffness → internal forces

[nb: introduction of the quasi-static response $\mathbf{x}_{qs}(t) = \mathbf{K}^{-1}\mathbf{p}(t)$]

The Equation Of Motion IV

Two different nomenclatures for two different versions of the equation of the motion:

Multi degree-of-freedom systems (matrix version, M-DOF)

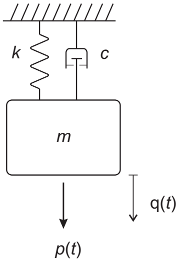
$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{p}(t) \tag{6}$$

Single degree-of-freedom systems (scalar version, S-DOF)

$$m\ddot{q}(t)+c\dot{q}(t)+kq(t)=p(t) \tag{7}$$

Discuss the importance to understand the behavior of the S-DOF system before tackling more complex M-DOF systems.

Single-Degree-Of-Freedom Oscillator I



$$\begin{cases} f_D(t) = c\,\dot{q}(t) \\ f_S(t) = k\,q(t) \end{cases}$$

$$\Rightarrow f_{tot} = p(t) - k\,q(t) - c\,\dot{q}(t)$$

Newton's law:

$$m\ddot{q}(t) = f_{tot} \quad \Rightarrow \quad m\ddot{q}(t) + c\dot{q}(t) + kq(t) = p(t) \tag{8}$$

[nb: connection with a real structure]

Single-Degree-Of-Freedom Oscillator II

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = p(t)$$

The structural behavior is characterized by 3 parameters: m , c and k

One can also write

$$\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t) = \frac{p(t)}{m}$$

where

$$\omega_1 = \sqrt{\frac{k}{m}} = \frac{2\pi}{T_1} = 2\pi f_1$$

$$\xi_1 = \frac{c}{2m\omega_1} = \frac{c}{2\sqrt{km}}$$

Equivalently, the structural properties are characterized by three other parameters: m, ω_1, ξ_1.

The Mathematician's Solution

$$\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t) = \frac{p(t)}{m}$$

The solution of a 2nd order ODE, non-homogenous, with constant coefficients is obtained by:

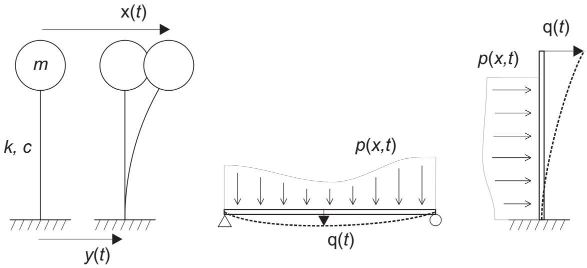
- establishing *the* general solution of the homogeneous equation $q_h(t)$ (with 2 constants of integration)
- finding *one* particular solution $q_p(t)$
- the solution then reads $q = q_h + q_p$
- constants A and B introduced in $q_h(t)$ are **then** determined from initial conditions

Analytical solutions are known is some cases only → need to:

- understand the physics on the basis of these simple analytical solutions
- develop numerical methods to treat more complex problems

Exercises

1. Express the equation of motion of a single-degree-of-freedom system excited by its support (i.e. seismic excitation).
2. Express the lumped equation of motion of a beam resting on two supports (assume that the deformed shape is sinusoidal)
3. Express the lumped equation of motion of a tower (assume that the deformed shape is quadratic/cubic/quartic)



Motivation

Solution of

$$\ddot{q}(t) + 2\omega_1 \xi_1 \dot{q}(t) + \omega_1^2 q(t) = \frac{p(t)}{m} \tag{9}$$

with $q(0) = 0$; $\dot{q}(0) = 0$?

Class of problem: Initial Value Problem (IVP)

Discuss case of an earthquake loading or complex loading

Satisfy the equation of motion at certain time instants t_0, t_1, ... *only* and assume a particular response in between.
nb: most usually $t_i = t_0 + i\Delta t$ (constant time step).

Motivation

Solution of

$$\ddot{q}(t) + 2\omega_1 \xi_1 \dot{q}(t) + \omega_1^2 q(t) = \frac{p(t)}{m} \tag{9}$$

with $q(0) = 0$; $\dot{q}(0) = 0$?

Class of problem: Initial Value Problem (IVP)

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Satisfy the equation of motion at certain time instants t_0, t_1, ... *only* and assume a particular response in between.
nb: most usually $t_i = t_0 + i\Delta t$ (constant time step).

Comparison with a Standard Problem

Standard Problem

$$\frac{d}{dt}\mathbf{y} = \dot{\mathbf{y}} = f(\mathbf{y}, t)$$

Equation of motion

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = p(t)$$

Let $x_1 = q$, $x_2 = \dot{q}$,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{1}{m}(p - c\,x_2 - k\,x_1) \end{pmatrix}$$

same formulation → possible to use the same methods as those that are used to solve the standard problem

General Facts

One way to classify methods:

- explicit methods (simple recurrence from time step to time step)

$$y_{t+\Delta t} = \text{fct}\left(y_t, y_{t-\Delta t}, \ldots\right)$$

- implicit methods (requires iteration inside a step)

$$y_{t+\Delta t} = \text{fct}\left(y_{t+\Delta t}, y_t, y_{t-\Delta t}, \ldots\right)$$

2

Another way to classify methods:

- methods based on approximations of the derivatives
- methods based on approximations of integrals

Examples I

Solution of $dy/dt = f(y,t)$

Method 1

1. Equation at time t:

$$\left.\frac{dy}{dt}\right|_t = f(y_t, t)$$

2. Approx. of the derivative (forward)

$$\left.\frac{dy}{dt}\right|_t \simeq \frac{y_{t+\Delta t} - y_t}{\Delta t}$$

Algortithm:

$$\frac{y_{t+\Delta t} - y_t}{\Delta t} = f(y_t, t) \quad \Rightarrow \quad y_{t+\Delta t} = y_t + \Delta t\, f(y_t, t)$$

$\rightarrow$ simple recurrence to obtain $y_{t+\Delta t}$ (explicit method)

Examples II
Solution of $dy/dt = f(y,t)$

Method 2

1. Equation at time $t + \Delta t$:

$$\left.\frac{dy}{dt}\right|_{t+\Delta t} = f\left(y_{t+\Delta t}, t+\Delta t\right)$$

2. Approx. of the derivative (backward)

$$\left.\frac{dy}{dt}\right|_{t+\Delta t} \simeq \frac{y_{t+\Delta t} - y_t}{\Delta t}$$

Algorithm:

$$\frac{y_{t+\Delta t} - y_t}{\Delta t} = f\left(y_{t+\Delta t}, t+\Delta t\right)$$

$\Rightarrow$ iterations are necessary to obtain $y_{t+\Delta t}$ (implicit method)

Examples III

Solution of $dy/dt = f(y,t)$

Method 3

Approx. of the integral (trapeze rule)

$$y_{t+\Delta t} = y_t + \int_t^{t+\Delta t} f(y,t)\,\mathrm{dt} \simeq y_t + \frac{f(y_{t+\Delta t}, t+\Delta t) + f(y_t, t)}{2}\Delta t$$

$\Rightarrow$ iterations to obtain $y_{t+\Delta t}$ (implicit method)

Method 4

Approx. of the integral (rectangle rule)

$$y_{t+\Delta t} = y_t + \int_t^{t+\Delta t} f(y,t)\,\mathrm{dt} \simeq y_t + f(y_t, t)\,\Delta t$$

$\Rightarrow$ same as method 1 (explicit)

How to select a method ?

Numerical methods introduce (spuriously!?)

- artificial (numerical) damping
- phase shift
- period alteration

Reasons

- troncature (cf. estimation of derivative .vs. Talyor series)
- transmission of errors from step to step (depending on the algorithm)

The time step has to be chosen in such a way to capture the fast variations of the response (natural period) and of the loading [use at least 10 points at least to represent one period of a sine

Contents

Central Difference Method I

The equation of motion is satisfied at time t

$$m\ddot{q}_t + c\dot{q}_t + kq_t = p_t$$

By central difference, the second derivative is approximated as

$$\ddot{q}_t \simeq \frac{q_{t-\Delta t} - 2q_t + q_{t+\Delta t}}{\Delta t^2} + \mathcal{O}\left(\Delta t^2\right)$$

so that

$$\frac{m}{\Delta t^2}\left(q_{t-\Delta t} - 2q_t + q_{t+\Delta t}\right) + c\dot{q}_t + kq_t = p_t$$

$$\Rightarrow q_{t+\Delta t} = \frac{\Delta t^2}{m}\left(p_t - c\dot{q}_t - kq_t\right) + 2q_t - q_{t-\Delta t}$$

i.e. an expression of $q_{t+\Delta t}$ as a function of q_t, $\dot{q}_t$ and $q_{t-\Delta t}$.

Eliminate $q_{t-\Delta t}$ by considering

$$\dot{q}_t \simeq \frac{q_{t+\Delta t} - q_{t-\Delta t}}{2\Delta t} + \mathcal{O}\left(\Delta t^2\right) \Rightarrow q_{t-\Delta t} = q_{t+\Delta t} - 2\Delta t\,\dot{q}_t \qquad (10)$$

so that

$$q_{t+\Delta t} = q_t + \Delta t\,\dot{q}_t + \frac{\Delta t^2}{2m}\left(p_t - c\dot{q}_t - kq_t\right) \qquad (11)$$

i.e. an expression of $q_{t+\Delta t}$ as a function of q_t, $\dot{q}_t$.

Central Difference Method II

Two possible closures (to close the recurrence)

Option 1: find/express $\dot{q}_{t+\Delta t}$ as a function of q_t, $\dot{q}_t$
Calculation of the velocity as (hyp: average velocity is obtained by the finite difference of positions):

$$\frac{\dot{q}_{t+\Delta t}+\dot{q}_t}{2}=\frac{q_{t+\Delta t}-q_t}{\Delta t}+\mathcal{O}(\Delta t)$$

or

$$\dot{q}_{t+\Delta t}=2\frac{q_{t+\Delta t}-q_t}{\Delta t}-\dot{q}_t \qquad (12)$$

In practice: starting from q_0 and $\dot{q}_0$, iterations are performed with (11) and (12).
(NB: this option requires a supplementary hypothesis)

Central Difference Method III

Option 2: eliminate $\dot{q}_t$ from (11) and obtain $q_{t+\Delta t}$ as a function of q_t, $q_{t-\Delta t}$.
Insert in (11) the expression of $\dot{q}_t$ given by (10):

$$q_{t+\Delta t} = q_t + \Delta t \frac{q_{t+\Delta t} - q_{t-\Delta t}}{2\Delta t} + \frac{\Delta t^2}{2m}\left(p_t - c\frac{q_{t+\Delta t} - q_{t-\Delta t}}{2\Delta t} - kq_t\right) \qquad (13)$$

$$\Leftrightarrow \left(\frac{m}{\Delta t^2} + \frac{c}{2\Delta t}\right) q_{t+\Delta t} = p_t + \left(\frac{2m}{\Delta t^2} - k\right) q_t - \left(\frac{m}{\Delta t^2} - \frac{c}{2\Delta t}\right) q_{t-\Delta t} \qquad (14)$$

Startup: $\dot{q}_0 = \frac{q_0 - q_{-\Delta t}}{\Delta t} \Rightarrow q_{-\Delta t} = q_0 - \Delta t\, \dot{q}_0$.

In practice: after determination of $q_{-\Delta t}$, iterate with (14).

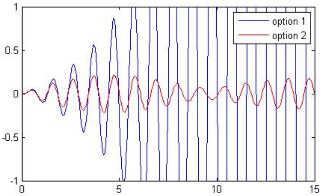
Central Difference Method IV

Characteristics Option 1:

- Explicit method
- Unstable

Characteristics Option 2:

- Explicit method
- Limit in the stability of the algorithm: unstable if Δt too large (not really problematic for SDOF systems but well for MDOF systems)



Example with: $m = 1$kg , $f = 1$Hz, $\xi = 1\%$

$p = \sin 2\pi f_0 t$, with $f_0 = 0.9$Hz

Exploratory Exercises

1. Implement both versions of the central difference method and play around with parameters.
2. Validate the implementation by comparing your results with those of the previous slide.
   (Numerical values: $m = 1$kg , $f = 1$Hz, $\xi = 1\%$, $p = \sin 2\pi f_0 t$, with $f_0 = 0.9$Hz)
3. Observe that
   3.1 option 1 provides an unbounded response (no matter the timestep)
   3.2 option 2 provides an unbounded response if the timestep is slightly above a critical value
4. Explore the features of the responses for several values of the problem parameters, in order to determine the critical value of the timestep (option 2 only)

Constant Acceleration Method I

The method is based on approximation of integrals:

$$\dot{q}_{t+\Delta t} = \dot{q}_t + \int_t^{t+\Delta t} \ddot{q}(t)\,dt \quad ; \quad q_{t+\Delta t} = q_t + \int_t^{t+\Delta t} \dot{q}(t)\,dt$$

Constant (average) acceleration hypothesis between t and $t+\Delta t$:

$$\ddot{q}(t+\tau) = \frac{\ddot{q}_t + \ddot{q}_{t+\Delta t}}{2}$$

(nb: $\ddot{q}_{t+\Delta t}$ is unknown). Hence,

$$\dot{q}(t+\tau) = \dot{q}_t + \frac{\ddot{q}_t + \ddot{q}_{t+\Delta t}}{2}\tau$$

$$q(t+\tau) = q_t + \dot{q}_t\tau + \frac{\ddot{q}_t + \ddot{q}_{t+\Delta t}}{2}\frac{\tau^2}{2}$$

(time evolution of the velocity and position inside a time step: $\tau \in [0;\Delta t]$)

Constant Acceleration Method II

At the end of the time step,

$$\dot{q}_{t+\Delta t} = \dot{q}_t + \Delta t \frac{\ddot{q}_t + \ddot{q}_{t+\Delta t}}{2} \tag{15}$$

$$q_{t+\Delta t} = q_t + \dot{q}_t \Delta t + \frac{\ddot{q}_t + \ddot{q}_{t+\Delta t}}{2} \frac{\Delta t^2}{2} \tag{16}$$

(as a function of $\ddot{q}_{t+\Delta t}$ unknown).
Substitue (15) and (16) in the equation of motion at time $t + \Delta t$, then solve for $\ddot{q}_{t+\Delta t}$.

Linear Acceleration Method I

The method is based on approximation of integrals:

$$\dot{q}_{t+\Delta t} = \dot{q}_t + \int_t^{t+\Delta t} \ddot{q}(t)\,dt \quad ; \quad q_{t+\Delta t} = q_t + \int_t^{t+\Delta t} \dot{q}(t)\,dt$$

Linear acceleration hypothesis between t and $t+\Delta t$:

$$\ddot{q}(t+\tau) = \ddot{q}_t + \frac{\ddot{q}_{t+\Delta t} - \ddot{q}_t}{\Delta t}\tau$$

(nb: $\ddot{q}_{t+\Delta t}$ is unknown). Hence,

$$\dot{q}(t+\tau) = \dot{q}_t + \ddot{q}_t\tau + \frac{\ddot{q}_{t+\Delta t} - \ddot{q}_t}{\Delta t}\frac{\tau^2}{2}$$

$$q(t+\tau) = q_t + \dot{q}_t\tau + \ddot{q}_t\frac{\tau^2}{2} + \frac{\ddot{q}_{t+\Delta t} - \ddot{q}_t}{\Delta t}\frac{\tau^3}{6}$$

Linear Acceleration Method II

At the end of the time step,

$$\dot{q}_{t+\Delta t} = \dot{q}_t + \ddot{q}_t \Delta t + \frac{\ddot{q}_{t+\Delta t} - \ddot{q}_t}{2} \Delta t = \dot{q}_t + \frac{\ddot{q}_{t+\Delta t} + \ddot{q}_t}{2} \Delta t \quad (17)$$

$$q_{t+\Delta t} = q_t + \dot{q}_t \Delta t + \ddot{q}_t \frac{\Delta t^2}{2} + \frac{\ddot{q}_{t+\Delta t} - \ddot{q}_t}{6} \Delta t^2 = q_t + \dot{q}_t \Delta t + \left(\frac{\ddot{q}_t}{3} + \frac{\ddot{q}_{t+\Delta t}}{6} \right) \Delta t^2 \quad (18)$$

Substitute (17) and (18) in the equation of motion at time $t + \Delta t$, then solve for $\ddot{q}_{t+\Delta t}$.

Newmark Methods I

Constant Acceleration	Linear Acceleration
$(\alpha = 1/4,\ \delta = 1/2)$	$(\alpha = 1/6,\ \delta = 1/2)$
$\dot{q}_{t+\Delta t} = \dot{q}_t + \Delta t \frac{\ddot{q}_t + \ddot{q}_{t+\Delta t}}{2}$	$\dot{q}_{t+\Delta t} = \dot{q}_t + \frac{\ddot{q}_{t+\Delta t} + \ddot{q}_t}{2} \Delta t$
$q_{t+\Delta t} = q_t + \dot{q}_t \Delta t + \frac{\ddot{q}_t + \ddot{q}_{t+\Delta t}}{2} \frac{\Delta t^2}{2}$	$q_{t+\Delta t} = q_t + \dot{q}_t \Delta t + \left(\frac{\ddot{q}_t}{3} + \frac{\ddot{q}_{t+\Delta t}}{6} \right) \Delta t^2$

General formalism of Newmark methods (nb: $\ddot{q}_{t+\Delta t}$ is unknown)

$$\dot{q}_{t+\Delta t} = \dot{q}_t + \left[(1-\delta)\, \ddot{q}_t + \delta \ddot{q}_{t+\Delta t} \right] \Delta t \tag{19}$$

$$q_{t+\Delta t} = q_t + \dot{q}_t \Delta t + \left[\left(\frac{1}{2} - \alpha \right) \ddot{q}_t + \alpha \ddot{q}_{t+\Delta t} \right] \Delta t^2 \tag{20}$$

Plus equation of motion at time $t + \Delta t$:

$$m \ddot{q}_{t+\Delta t} + c \dot{q}_{t+\Delta t} + k q_{t+\Delta t} = p_{t+\Delta t} \tag{21}$$

Newmark Methods II

Implicit scheme ($\Rightarrow$ requires *a priori* iterations to find $\ddot{q}_{t+\Delta t}$).

Method:

- start from an iterate $\ddot{q}_{t+\Delta t}$,
- use (19) and (20) to obtain $\dot{q}_{t+\Delta t}$ and $q_{t+\Delta t}$,
- use the equation of motion (21) to obtain a new value of the iterate $\ddot{q}_{t+\Delta t}$. Loop.

Conversion into an explicit scheme (possible in case of linear system)

$$\begin{bmatrix} 0 & 1 & -\delta\Delta t \\ 1 & 0 & -\alpha\Delta t^2 \\ k & c & m \end{bmatrix} \begin{pmatrix} q_{t+\Delta t} \\ \dot{q}_{t+\Delta t} \\ \ddot{q}_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \dot{q}_t + (1-\delta)\,\ddot{q}_t\Delta t \\ q_t + \dot{q}_t\Delta t + \left(\frac{1}{2} - \alpha\right)\ddot{q}_t\Delta t^2 \\ p_{t+\Delta t} \end{pmatrix}$$

Newmark Methods III

Notice this latter equation may also be written

$$\begin{bmatrix} 0 & 1 & -\delta\Delta t \\ 1 & 0 & -\alpha\Delta t^2 \\ k & c & m \end{bmatrix} \begin{pmatrix} q_{t+\Delta t} \\ \dot{q}_{t+\Delta t} \\ \ddot{q}_{t+\Delta t} \end{pmatrix} = \begin{bmatrix} 0 & 1 & (1-\delta)\Delta t \\ 1 & \Delta t & \left(\frac{1}{2}-\alpha\right)\Delta t^2 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} q_t \\ \dot{q}_t \\ \ddot{q}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_{t+\Delta t} \end{pmatrix}$$

$$\begin{bmatrix} 0 & 1 & -\delta \\ 1 & 0 & -\alpha \\ \beta^2 & 2\beta\xi_1 & 1 \end{bmatrix} \begin{pmatrix} q_{t+\Delta t} \\ \Delta t\dot{q}_{t+\Delta t} \\ \Delta t^2\ddot{q}_{t+\Delta t} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 1-\delta \\ 1 & 1 & \frac{1}{2}-\alpha \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} q_t \\ \Delta t\dot{q}_t \\ \Delta t^2\ddot{q}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_{t+\Delta t}/m \end{pmatrix}$$

with $\beta = \omega_1 \Delta t$

Newmark Methods IV

Solve for $q_{t+\Delta t}$ (for instance)

$$\left(\frac{1}{\alpha\Delta t^2}m+\frac{\delta}{\alpha\Delta t}c+k\right)q_{t+\Delta t}=p_{t+\Delta t}$$
$$+c\left(\frac{\delta}{\alpha\Delta t}q_t+\left(\frac{\delta}{\alpha}-1\right)\dot{q}_t+\frac{\Delta t}{2}\left(\frac{\delta}{\alpha}-2\right)\ddot{q}_t\right)$$
$$+m\left(\frac{1}{\alpha\Delta t^2}q_t+\frac{1}{\alpha\Delta t}\dot{q}_t+\left(\frac{1}{2\alpha}-1\right)\ddot{q}_t\right)$$

$$\Leftrightarrow k_F\,q_{t+\Delta t}=p_F$$

then

$$\dot{q}_{t+\Delta t}=\frac{\delta}{\alpha\Delta t}\left(q_{t+\Delta t}-q_t\right)+\left(1-\frac{\delta}{\alpha}\right)\dot{q}_t+\Delta t\left(1-\frac{\delta}{2\alpha}\right)\ddot{q}_t$$
$$\ddot{q}_{t+\Delta t}=\frac{1}{\alpha\Delta t^2}\left(q_{t+\Delta t}-q_t\right)-\frac{1}{\alpha\Delta t}\dot{q}_t-\left(\frac{1}{2\alpha}-1\right)\ddot{q}_t$$

Contents

Stability of Numerical Methods I

Write down the equations of the numerical scheme under the canonical form

$$\mathbf{x}_{t+\Delta t} = \mathbf{A}\,\mathbf{x}_t + \mathbf{L}\,\mathbf{r}_t$$

where $\mathbf{x}_t$ is composed of displacements, velocities and/or accelerations and $\mathbf{r}_t$ depends on p_t, $p_{t-\Delta t}$, etc.

Central Difference, option 1

The equations of the Central Difference (11)-(12) are written

$$q_{t+\Delta t} = q_t + \Delta t\,\dot{q}_t + \frac{\Delta t^2}{2m}\left(p_t - c\dot{q}_t - kq_t\right)$$

$$\dot{q}_{t+\Delta t} = \frac{2}{\Delta t}\left(q_{t+\Delta t} - q_t\right) - \dot{q}_t = \dot{q}_t + \frac{\Delta t}{m}\left(p_t - c\dot{q}_t - kq_t\right)$$

i.e.

$$\begin{pmatrix} q_{t+\Delta t} \\ \Delta t\,\dot{q}_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} 1-\frac{(\omega_1\Delta t)^2}{2} & 1-\frac{\xi_1\omega_1\Delta t}{2} \\ -(\omega_1\Delta t)^2 & 1-2\xi_1\omega_1\Delta t \end{pmatrix} \begin{pmatrix} q_t \\ \Delta t\,\dot{q}_t \end{pmatrix} + \begin{pmatrix} \frac{\Delta t^2}{2m} \\ \frac{\Delta t^2}{m} \end{pmatrix} p_t$$

Central Difference, option 2

The equations of the central difference (13)-(14) are written

$$\ddot{q}_t = \frac{1}{\Delta t^2}\left(q_{t-\Delta t} - 2q_t + q_{t+\Delta t}\right)$$

$$\dot{q}_t = \frac{1}{2\Delta t}\left(q_{t+\Delta t} - q_{t-\Delta t}\right)$$

Introduced in $\ddot{q}_t + 2\omega_1\xi_1\dot{q}_t + \omega_1^2 q_t = \frac{p_t}{m}$, we get

$$\begin{pmatrix} q_{t+\Delta t} \\ q_t \end{pmatrix} = \begin{pmatrix} \frac{2-(\omega_1\Delta t)^2}{1+\xi_1(\omega_1\Delta t)} & -\frac{1-\xi_1(\omega_1\Delta t)}{1+\xi_1(\omega_1\Delta t)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_t \\ q_{t-\Delta t} \end{pmatrix} + \begin{pmatrix} \frac{\Delta t^2}{1+\xi_1(\omega_1\Delta t)} \\ 0 \end{pmatrix} p_t$$

Stability of Numerical Methods III

The stability is studied on the free response ($\mathbf{r} \equiv 0$), so that

$$\mathbf{x}_{t+\Delta t} = \mathbf{A}\mathbf{x}_t = \mathbf{A}^2\mathbf{x}_{t-\Delta t} = \ldots = \mathbf{A}^{n+1}\mathbf{x}_0.$$

The response is bounded for $n \to \infty$ iif $\mathbf{A}^n$ remains bounded.

The n^{th} power of a matrix is computed via its spectral decomposition. Let

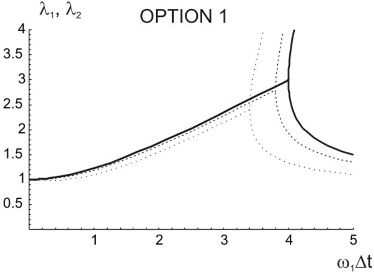
$$\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}^{-1},$$

then (Λ contains the eigen values of $\mathbf{A}$)

$$\mathbf{A}^n = \mathbf{P}\Lambda^n\mathbf{P}^{-1},$$

which indicates that $\mathbf{A}^n$ is bounded iif $\max_i |\Lambda_i| \leq 1$.
If $\max_i |\Lambda_i| > 1$, then $\mathbf{A}^n \to \infty$: undesired amplification of perturbations
If $\max_i |\Lambda_i| = 1$: initial perturbations remain undefinitely
If $\max_i |\Lambda_i| < 1$, then $\mathbf{A}^n \to 0$: artificial damping resulting from the numerical method

Stability of Numerical Methods IV



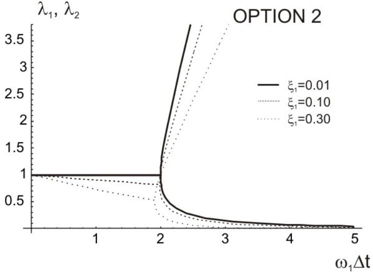
Central Difference, option 1 ($\xi_1 = 0$)

$$\mathbf{A} = \begin{pmatrix} 1 - \frac{(\omega_1 \Delta t)^2}{2} & 1 \\ -(\omega_1 \Delta t)^2 & 1 \end{pmatrix}$$

$$\lambda_{1,2} = 1 - \frac{(\omega_1 \Delta t)^2}{4} \pm \omega_1 \Delta t \sqrt{\left(\frac{\omega_1 \Delta t}{4}\right)^2 - 1}$$

The integration scheme is unstable in any case.

Stability of Numerical Methods V



Central Difference, option 2 ($\xi_1 = 0$)

$$\mathbf{A} = \begin{pmatrix} 2-(\omega_1\Delta t)^2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda_{1,2} = 1 - \frac{(\omega_1\Delta t)^2}{2} \pm \sqrt{\left(1 - \frac{(\omega_1\Delta t)^2}{2}\right)^2 - 1}$$

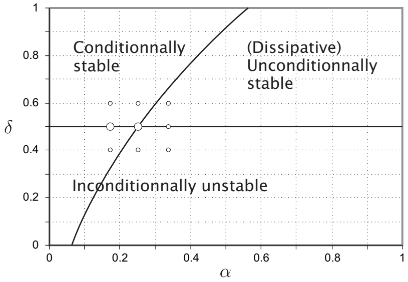
The integration scheme is stable if $\omega_1\Delta t \leq 2$, i.e. $\Delta t \leq T_1/\pi$

$\rightarrow$ conditionally stable

Stability of Numerical Methods VI

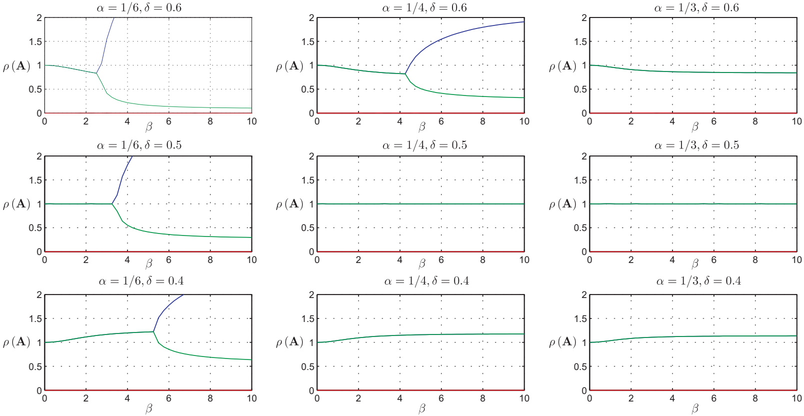
Stability of Nemark algorithms

Limits of stability for the undamped oscillator



Linear acceleration: limits of unstable domain (!)
Constant acceleration: spectral radius = 1, unconditionnally stable & non-dissipative

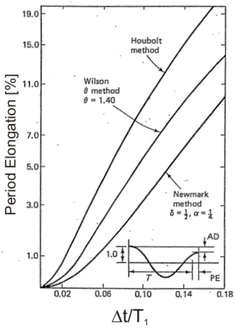
Stability of Newmark algorithms: spectral radius



Accuracy of Numerical Methods I

Besides being stable, it is expected that a numerical method be accurate. Two major defects are:

- amplitude degradation, numerical damping (cf. stability) - nb: sometimes useful
- period elongation



Example here for

$$\ddot{q} + \omega_1^2 q = 0 \quad q(0) = 1; \dot{q}(0) = 0$$

(exact solution: $q = \cos 2\pi \frac{t}{T_1}$)

→ limitations on the time step

Suggested Exercises I

Numerical integrators

1. With the help of a series of well-chosen numerical simulations, check the accuracy of the central difference algorithm, in particular the period elongation.
2. With the help of a series of well-chosen numerical simulations, check the stability and accuracy of Newmark's algorithm. Validate your findings with the computation of the spectral radius of the iteration matrix.
3. Do the same for other integration schemes: HHT, Houbolt, Wilson, Bathe, etc.

Suggested Exercises II

Structural analysis

1. An object falls at the mid-span of a 4-m span simply supported beam. It generates a force assumed to be expressed as

$$F(t) = F_0 \frac{\sqrt{t/t_0}}{1+(t/t_0)^2}$$

with $F_0 = 2000N$ and $t_0 = 0.1$s. Determine the maximum bending moment/stress in the beam made of pine timber (cross-section of 6.5×18).

2. A single bay frame with pinned end beam is subject to a harmonic excitation at the top. (to be developed).

Contents

Multi Degree-Of-Freedom Structures I

Two different nomenclatures for two different versions of the equation of the motion:

Multi-degree of freedom systems (matrix version, M-DOF)

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{p}(t)$$

Systems with 1 degree-of-freedom (scalar version, S-DOF)

$$m\ddot{q}(t)+c\dot{q}(t)+kq(t)=p(t)$$

The solution of

$$\dot{y}=a\,y \quad \text{is} \quad y=C_0 e^{at}$$

By extension, the solution of

$$\dot{\mathbf{y}}=\mathbf{A}\,\mathbf{y} \quad \text{is} \quad \mathbf{y}=\mathbf{C}_0 e^{\mathbf{A}t}$$

Extension to the MDOF case of the solutions obtained in the SDOF case (upon condition of the definition of the exponential of a matrix...).

Multi Degree-Of-Freedom Structures II

1-DOF system:

$$\left(\frac{1}{\alpha\Delta t^2}m+\frac{\delta}{\alpha\Delta t}c+k\right)q_{t+\Delta t}=p_{t+\Delta t}+c\left(\frac{\delta}{\alpha\Delta t}q_t+\left(\frac{\delta}{\alpha}-1\right)\dot{q}_t+\frac{\Delta t}{2}\left(\frac{\delta}{\alpha}-2\right)\ddot{q}_t\right)$$
$$+m\left(\frac{1}{\alpha\Delta t^2}q_t+\frac{1}{\alpha\Delta t}\dot{q}_t+\left(\frac{1}{2\alpha}-1\right)\ddot{q}_t\right)$$

M-DOF system:

$$\left(\frac{1}{\alpha\Delta t^2}\mathbf{M}+\frac{\delta}{\alpha\Delta t}\mathbf{C}+\mathbf{K}\right)\mathbf{x}_{t+\Delta t}=\mathbf{p}_{t+\Delta t}+\mathbf{C}\left(\frac{\delta}{\alpha\Delta t}\mathbf{x}_t+\left(\frac{\delta}{\alpha}-1\right)\dot{\mathbf{x}}_t+\frac{\Delta t}{2}\left(\frac{\delta}{\alpha}-2\right)\ddot{\mathbf{x}}_t\right)$$
$$+\mathbf{M}\left(\frac{1}{\alpha\Delta t^2}\mathbf{x}_t+\frac{1}{\alpha\Delta t}\dot{\mathbf{x}}_t+\left(\frac{1}{2\alpha}-1\right)\ddot{\mathbf{x}}_t\right)$$

- Discuss optimal choice of Newmark parameters
- [example of the multi-storey building - with undesirable high-frequency bracing vibrations]

Exercises

1.

Section III: Single Degree-Of-Freedom Systems

Learning Outcomes:

- understand the free response of an SDOF system
- recognize the signature of an SDOF system in the time and frequency domains
- get acquaintance with the Dynamic Amplification Factor, stationary and transient phases of the response
- at the end of this lecture, you should be able to discriminate between time and frequency domain for the solution of a given problem

Contents

Free Vibrations: Exploratory Exercises

1. Observe the free response of a single degree-of-freedom system, i.e.
$$\ddot{q}(t) + 2\xi\omega_1\dot{q}(t) + \omega_1^2 q(t) = 0$$
with $\omega_1 = 1$ rad/s, $\dot{q}(0) = 0$ and various values of ξ , $q(0)$.
[nb: Superimpose the responses on the same plot]

2. How much time does it take for the vibrations to damp out ?
Guess how this time is related to the system parameters.

Free Vibrations I

Free vibrations ≡ no external loading ($p(t)=0$). The equation reads

$$\ddot{q}(t)+2\omega_1\xi_1\dot{q}(t)+\omega_1^2 q(t)=0$$

Calculus: $q_p=0 \rightarrow$ determination of q_h and integration constants

- write and solve the characteristic equation (in z):

$$z^2+2\omega_1\xi_1 z+\omega_1^2=0,$$

$$z_1=-\omega_1\left(\xi_1+\imath\sqrt{1-\xi_1^2}\right) \quad ; \quad z_2=-\omega_1\left(\xi_1-\imath\sqrt{1-\xi_1^2}\right)$$

  (because $\xi_1 \leq 1$ in civil engineering applications. Usually $\xi_1 \sim 1\% \ll 1$)

- write the general solution (if $z_1 \neq z_2$):

$$q=C_1 e^{z_1 t}+C_2 e^{z_2 t} \quad \text{or} \quad q=e^{-\xi_1\omega_1 t}\left(A\cos\omega_d t+B\sin\omega_d t\right)$$

- determine constants C_1 and C_2 (or A and B) from initial conditions.

Free Vibrations II

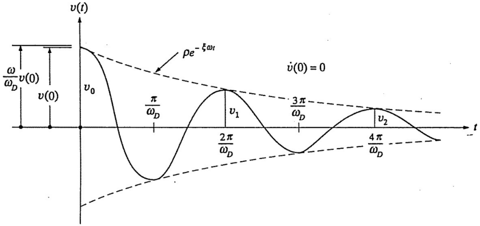
$$q(0) = q_0 \quad ; \quad \dot{q}(0) = \dot{q}_0. \tag{22}$$

So $A = q_0$ and $B = (\dot{q}_0 + \xi_1 \omega_1 q_0)/\omega_d$, and finally

$$q = e^{-\xi_1 \omega_1 t}\left(q_0 \cos \omega_d t + \frac{\dot{q}_0 + \xi_1 \omega_1 q_0}{\omega_d} \sin \omega_d t\right). \tag{23}$$

with $\omega_d = \omega_1 \sqrt{1-\xi^2}$

The free response is harmonic, with a circular frequency $\omega_d \simeq \omega_1$, and is modulated by a decreasing exponential



Free Vibrations III

1. The frequency of the free response motion $\omega_d = \omega_1\sqrt{1-\xi^2}$ is close to the natural frequency for slightly damped systems

Material	ξ_1	Material	ξ_1
Welded steel	0.1%-0.5%	Concrete	1%-2%
Bolted steel	0.5%-1%	Timber	2%-5%

→ meaning of the natural frequency

2. The envelope $e^{-\xi_1\omega_1 t} = e^{-\xi_1 2\pi n}$, with $n = t/T_1$ (modulation) is exponentially decreasing. The decrease just depends on ξ_1, when expressed in terms of the number of cycles.

3. Existence of a memory lag t_r
Observing that $e^{-\pi} = 4\%(\ll 1)$, we notice that the oscillation is damped out after a time t_r such that

$$-\pi = -\xi_1 2\pi \frac{t_r}{T_1} \qquad \rightarrow \qquad t_r = \frac{T_1}{2\xi_1} \tag{24}$$

Free Vibrations IV

The memory lag is the period of time during which the structure remembers perturbations (impacts, initial conditions, etc.)

4. Limit case for $\xi_1 \to 0$
The free response (23) regularly tends towards

$$q = q_0 \cos \omega_1 t + \frac{\dot{q}_0}{\omega_1} \sin \omega_1 t,$$

which shows that the amplitude of the undamped motion does not decrease.

5. Particular case for $\dot{q}_0 = 0$ (free launch)
The free response (23) then reads

$$q = q_0 e^{-\xi_1 \omega_1 t} \left(\cos \omega_d t + \frac{\xi_1 \omega_1}{\omega_d} \sin \omega_d t \right) \simeq q_0 e^{-\xi_1 \omega_1 t} \cos \omega_d t$$

Free Vibrations V

The free response is used to *identify* the natural period and the damping coefficient.

- relative maxima occur at times: $t_i = \frac{2i\pi}{\omega_d} \quad i = 0, 1, \ldots \rightarrow$ estimation of the natural frequency
- the successive maximum displacements are: $q_i = q_0 e^{-\xi_1 \omega_1 \frac{2i\pi}{\omega_d}}$
- the *logarithmic decrement* is defined as

$$\delta_1 = \ln \frac{q_i}{q_{i+1}} = \ln \frac{q_0 e^{-\xi_1 \omega_1 \frac{2i\pi}{\omega_d}}}{q_0 e^{-\xi_1 \omega_1 \frac{2(i+1)\pi}{\omega_d}}} = \frac{2\pi\xi_1}{\sqrt{1-\xi_1^2}},$$

$$\rightarrow \quad \xi_1 \simeq \frac{1}{2\pi} \ln \frac{q_i}{q_{i+1}} \quad \text{or} \quad \xi_1 \simeq \frac{1}{2n\pi} \ln \frac{q_i}{q_{i+n}}$$

Free Vibrations: Case Study I

Peterbos Footbridge

Maître de l'ouvrage : Bruxelles Mobilité-Direction, Gestion et Entretien des voiries
Adjudicataire : Association momentanée Groupe Verhaeren & Co et EMERGO NV

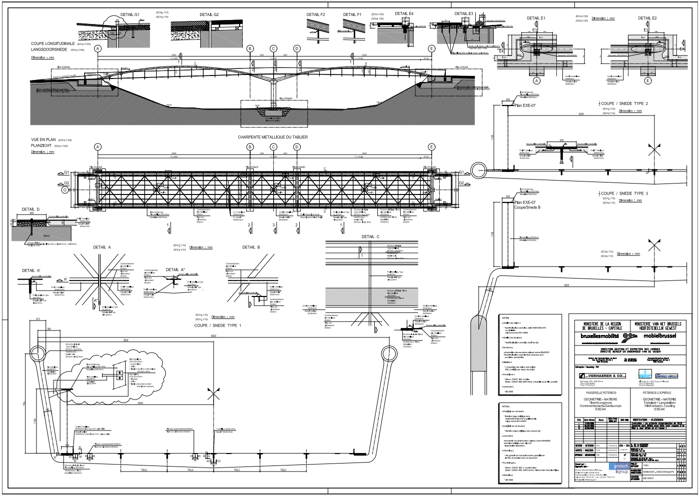
Bureau d'Etudes: BGroup, Bureau d'Etudes Greisch

Objectives of the study: determine the vibration amplitudes of this footbridge under pedestrian loading

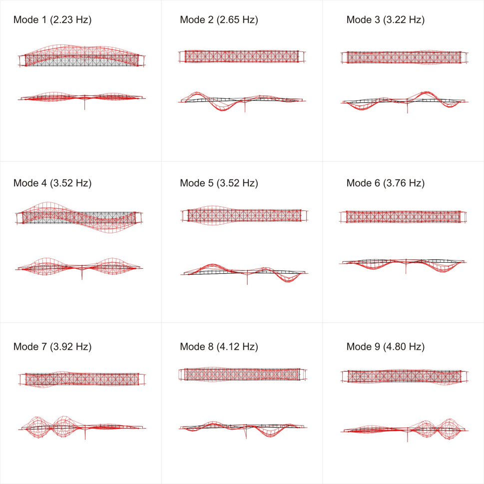
Step 1: Build finite element model of the structure
Step 2: Predict natural frequencies and mode shapes
Step 3: Validate with on-site measurement
Step 4: Perform finite element analysis

Free Vibrations: Case Study II



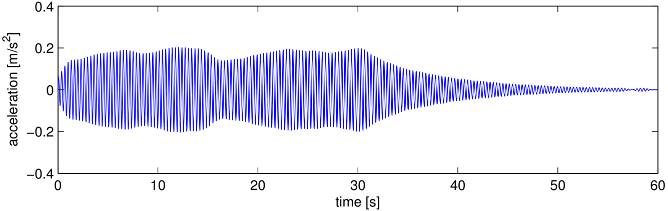
Free Vibrations: Case Study III

Predicted mode shapes



Free Vibrations: Case Study IV

Example of free decay response



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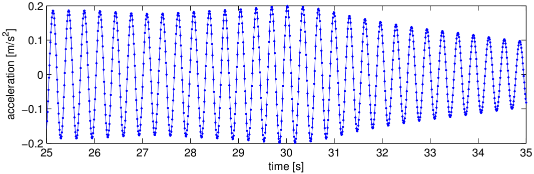
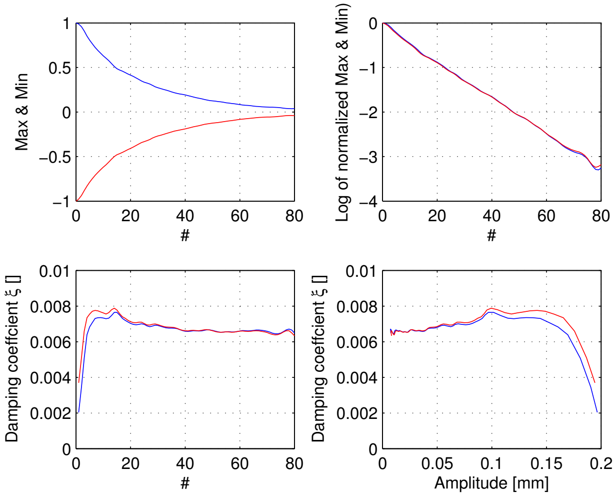
Free Vibrations: Case Study V



Free Vibrations: Case Study VI



Free Vibrations: Case Study VII

Mode	f_1 [Hz], computed	f_1 [Hz], measured	ξ_1	ε_{f_1}
1	2.23	2.10	?	+6%
2	2.65	2.79	1.6%	-5%
3	3.22	2.82	1.9%	+14%
4	3.52	3.20	0.55%	+10%
5	3.52	3.18	0.5%	+12%
6	3.76	3.88	0.7%	-3%
7	3.92	4.27	0.7%	-8%
8	4.12	4.35	0.5%	-5%
9	4.80	5.20	0.6%	-8%

Exercises

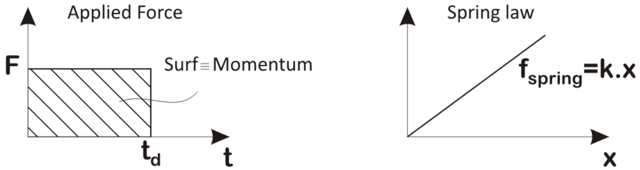
1. Identify the damping ratio from measured free-response accelerations.
2. Le poids W d'un bâtiment est de 900 kN et la réponse libre du bâtiment est étudiée en le relâchant à l'instant $t = 0$ depuis un déplacement de 3cm. Si le déplacement maximum après le premier retour dans la direction d'où on a réalisé le lâché libre est de 2.2 cm, à l'instant $t = 0.64$s, calculez (i) la raideur transversale k du bâtiment , (ii) la fréquence propre du bâtiment, (iii) le coefficient d'amortissement, (iv) la viscosité c du bâtiment.
3. Supposons que la masse et la raideur d'une structure soient égales à $m = 3,5.10^5 kg$ et $k = 7.10^6 N/m$. Si la structure est mise en vibrations libres avec les conditions initiales $x_0 = 1.78$cm and $\dot{x}_0 = 0.14$m/s, déterminez le déplacement et la vitesse après une second de vibration lorsque (i) $\xi = 0$ (pas d'amortissement), (ii) $\xi = 0.03$.

Impulsive Loading: Exploratory Exercises

1. Study the impulsive response of a single degree-of-freedom system, i.e.
$$\ddot{q}(t)+2\xi\omega_1\dot{q}(t)+\omega_1^2 q(t)=\begin{cases}F/m & \text{for } 0\leq t\leq t_d\\ 0 & \text{otherwise}\end{cases}$$
starting from rest position, with $m=1kg$, $F=1N$, $\omega_1=1\text{rad/s}$ and $\xi=0.01$.
2. Focus first on the case where $t_d \ll \frac{2\pi}{\omega_1}$. Compute the response for various values of t_d and observe the dynamic amplification factor (defined as the ratio of the maximum dynamic displacement to the maximum quasi-static displacement).
3. Focus then on the case where $t_d \gg \frac{2\pi}{\omega_1}$. Compute the response for various values of t_d.
4. Does the damping ratio affect that much the signature of the response?

Impulsive Loading I

Response under rectangular pulse



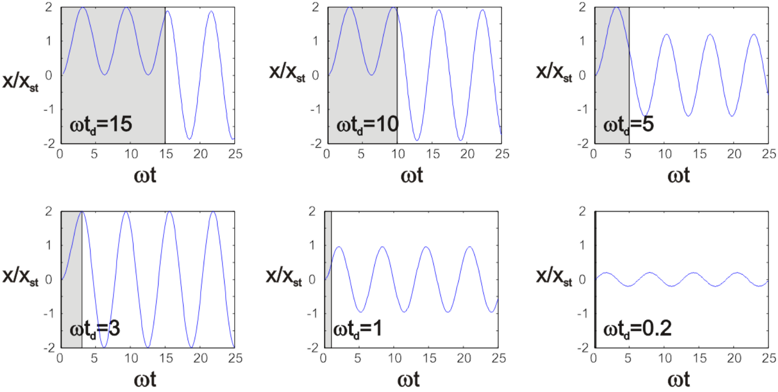
$$\ddot{q}+\omega_1^2 q = \begin{cases} F/m & \text{if } t < t_d \\ 0 & \text{if } t > t_d \end{cases} \tag{25}$$

$$q(t) = \frac{F}{k}\begin{cases} 1-\cos\omega_1 t & \text{if } t < t_d \\ \cos\left[\omega_1 (t-t_d)\right] - \cos\omega_1 t & \text{if } t > t_d \end{cases} \tag{26}$$

nb: $F/k = q_{static}$
nb II: $q(t)/q_{static}$ = dynamic amplification factor

Impulsive Loading II

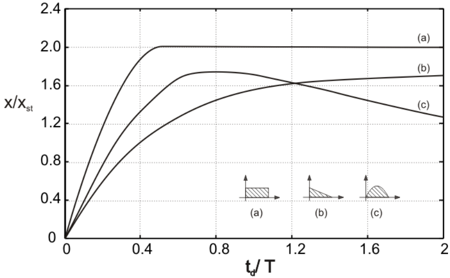
Response under a rectangular pulse with finite duration



Impulsive Loading III

Observations:

- $\frac{x}{x_{st}} = 2$ for $\omega_1 t_d \gg$, i.e. $t_d \gg T_1$ (Heaviside-like loading)
- $\frac{x}{x_{st}} \to 0$ for $\omega_1 t_d \ll$ (Dirac-like loading)



Impulsive Loading IV

Particular case $\omega_1 t_d \ll$

For $t_d \to 0$, the response (26) becomes

$$\begin{aligned} x(t) &= \frac{F}{k} \lim_{t_d \to 0} \left[\cos\left[\omega_1 (t - t_d)\right] - \cos \omega_1 t\right] \\ &= \frac{F}{k} \lim_{t_d \to 0} \left[2 \sin \frac{\omega_1 t_d}{2} \sin \omega_1 \left(t - \frac{t_d}{2}\right)\right] \\ &= \frac{F\, t_d \omega_1}{k} \sin \omega_1 t = \frac{I}{m\omega_1} \sin \omega_1 t \end{aligned}$$

When $t_d \ll T_1$, the momentum I governs the response, not the maximum force F !

Impulsive Loading V

Dirac Pulse Loading

The applied force is a Dirac function:

$$\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t) = \frac{I\delta(t)}{m}$$

with initial conditions at rest: $q(0) = 0$; $\dot{q}(0) = 0$.

NB: properties of a Dirac function:

- $\forall t \neq t_0 : \quad \delta(t - t_0) = 0$
- $\int_{-\infty}^{+\infty} f(t)\,\delta(t - t_0)\,dt = f(t_0)$
- units of $\delta(t)$= units of $\frac{1}{t}$
- example of Dirac function: $\delta(x) = \lim_{\sigma\to 0} \frac{e^{\frac{-x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$

Impulsive Loading VI

Integration along a short time window

$$\int_0^{\mathrm{dt}} \left(\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t)\right)\mathrm{dt} = \int_0^{\mathrm{dt}} \frac{I\delta(t)}{m}\mathrm{dt} = \frac{I}{m}$$

Limits for $\mathrm{dt} \to 0$,

$$\left.\begin{array}{c} \int_0^{\mathrm{dt}} \ddot{q}(t)\,\mathrm{dt} = \dot{q}(t)\rfloor_0^{dt} = \dot{q}(dt) \to \dot{q}(0^+) \\ \int_0^{\mathrm{dt}} \dot{q}(t)\,\mathrm{dt} = q(t)\rfloor_0^{dt} = q(dt) \to 0 \\ \int_0^{\mathrm{dt}} q(t)\,\mathrm{dt} \to 0 \end{array}\right\} \Rightarrow \quad \dot{q}(0^+) = \frac{I}{m}$$

The response to a Dirac impulse is thus a free response with initial conditions $q(0) = 0$, $\dot{q}(0^+) = \frac{I}{m}$:

$$q(t) \underset{\triangle}{=} I\,h(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \frac{I}{m\omega_d} e^{-\xi\omega_1 t} \sin\omega_d t & \text{for } t > 0 \end{cases}$$

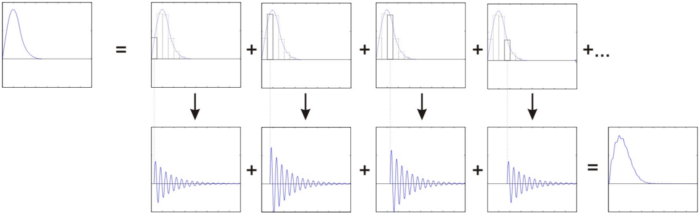
Arbitrary Loading I

By extension, the response to a unit pulse applied at time $t = \tau$ reads

$$h(t-\tau) = \begin{cases} 0 & \text{for } t \leq \tau \\ \frac{1}{m\omega_d} e^{-\xi\omega_1(t-\tau)} \sin\omega_d(t-\tau) & \text{for } t > \tau \end{cases}$$

[unitary impulsive response]

Any loading is just a sequence of impulses:



Arbitrary Loading II

Let us consider the pulse applied at time $t = \tau$:

$$dI = p(\tau)\, d\tau$$

The response to this pulse is obtained by multiplication by the unitary impulsive response $h(t-\tau)$:

$$dq = h(t-\tau)\, dI = h(t-\tau)\, p(\tau)\, d\tau$$

Considering now the sequence of pulses in $p(t)$:

$$q(t) = \int_0^{+\infty} h(t-\tau)\, p(\tau)\, d\tau$$

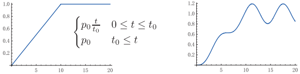
or

$$q(t) = \int_0^t \frac{p(\tau)}{m\omega_d} e^{-\xi\omega_1(t-\tau)} \sin\omega_d (t-\tau)\, d\tau$$

[Convolution integral ; Duhamel's integral]

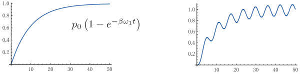
Arbitrary Loading III

$$\frac{p_0}{k}\left(1-\cos\omega_1 t\right)$$



$$\begin{cases} \frac{p_0}{k}\left(\frac{t}{t_0}-\frac{\sin\omega_1 t}{\omega_1 t_0}\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\left(1+\frac{\sin(\omega_1(t-t_0))-\sin(\omega_1 t)}{\omega_1 t_0}\right) & t_0 \le t \end{cases}$$

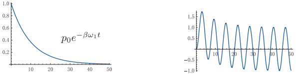
$$\frac{at}{k}\left(1-\frac{\sin\omega_1 t}{\omega_1 t}\right)$$



1

$$\frac{p_0}{k}\left(1-\frac{e^{-\beta\omega_1 t}+\beta\sin\omega_1 t+\beta^2\cos\omega_1 t}{1+\beta^2}\right)$$

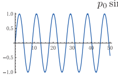
$$\frac{bt^2}{k}\left(1-2\frac{1-\cos\omega_1 t}{\omega_1 t^2}\right)$$

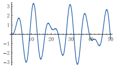


$$\frac{p_0}{k}\frac{e^{-\beta\omega_1 t}+\beta\sin\omega_1 t-\cos\omega_1 t}{1+\beta^2}$$

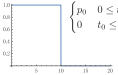
Arbitrary Loading IV

$p_0 \sin \beta\omega_1 t$

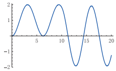




$$\frac{p_0}{k}\frac{\sin\beta\omega_1 t - \beta\sin\omega_1 t}{1-\beta^2}$$

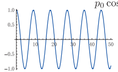


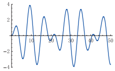
$$\begin{cases} p_0 & 0 \le t \le t_0 \\ 0 & t_0 \le t \end{cases}$$



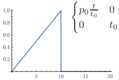
$$\begin{cases} \frac{p_0}{k}\left(1-\cos\omega_1 t\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\left(\cos\omega_1\left(t-t_0\right)-\cos\omega_1 t\right) & t_0 \le t \end{cases}$$

$p_0 \cos \beta\omega_1 t$

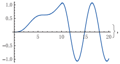




$$\frac{p_0}{k}\frac{\cos\beta\omega_1 t - \cos\omega_1 t}{1-\beta^2}$$

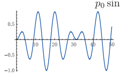


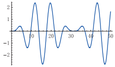
$$\begin{cases} p_0 \frac{t}{t_0} & 0 \le t \le t_0 \\ 0 & t_0 \le t \end{cases}$$



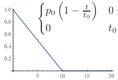
$$\begin{cases} \frac{p_0}{k}\frac{t}{t_0}\left(1-\frac{\sin\omega_1 t}{\omega_1 t}\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\left(\cos\omega_1\left(t-t_0\right)+\frac{\sin\omega_1(t-t_0)-\sin\omega_1 t}{\omega_1 t_0}\right) & t_0 \le t \end{cases}$$

$p_0 \sin \alpha\omega_1 t \sin \beta\omega_1 t$

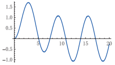




$$\frac{p_0}{k}\frac{\left(1-\alpha^2-\beta^2\right)\sin\alpha\omega_1 t\sin\beta\omega_1 t+2\alpha\beta\left(\cos\omega_1 t-\cos\alpha\omega_1 t\cos\beta\omega_1 t\right)}{\left(1-(\alpha+\beta)^2\right)\left(1-(\alpha-\beta)^2\right)}$$

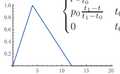


$$\begin{cases} p_0\left(1-\frac{t}{t_0}\right) & 0 \le t \le t_0 \\ 0 & t_0 \le t \end{cases}$$

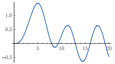


$$\begin{cases} \frac{p_0}{k}\left(1-\frac{t}{t_0}-\cos\omega_1 t+\frac{\sin\omega_1 t}{\omega_1 t_0}\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\left(-\cos\omega_1 t-\frac{\sin\omega_1(t-t_0)-\sin\omega_1 t}{\omega_1 t_0}\right) & t_0 \le t \end{cases}$$

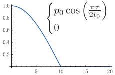
Arbitrary Loading V



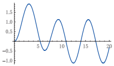
$$\begin{cases} p_0 \frac{t}{t_0} & 0 \le t \le t_0 \\ p_0 \frac{t_1 - t}{t_1 - t_0} & t_0 \le t \le t_1 \\ 0 & t_0 \le t \end{cases}$$



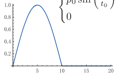
$$\begin{cases} \frac{p_0}{k}\left(\frac{t}{t_0} - \frac{\sin\omega_1 t}{\omega_1 t}\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\left(\frac{t_1 - t}{t_1 - t_0} - \frac{\sin\omega_1 t)}{\omega_1 t_0} - \frac{t_1}{t_0}\frac{\sin\omega_1 (t - t_0)}{\omega_1 (t_0 - t_1)}\right) & t_0 \le t \le t_1 \\ -\frac{p_0}{k}\frac{(t_0 - t_1)\sin\omega_1 t + t_1 \sin(\omega_1(t - t_0)) - t_0 \sin(\omega_1(t - t_1))}{t_0 \omega_1 (t_0 - t_1)} & t_0 \le t \end{cases}$$



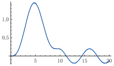
$$\begin{cases} p_0 \cos\left(\frac{\pi\tau}{2t_0}\right) & 0 \le t \le t_0 \\ 0 & t_0 \le t \end{cases}$$



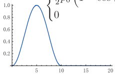
$$\begin{cases} \frac{p_0}{k}\frac{4\omega_1^2 t_0^2}{4\omega_1^2 t_0^2 - \pi^2}\left(\cos\frac{\pi t}{2t_0} - \cos\omega_1 t\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\frac{4\omega_1^2 t_0^2}{4\omega_1^2 t_0^2 - \pi^2}\left(-\frac{\pi}{2\omega_1 t_0}\sin\omega_1 (t - t_0) - \cos\omega_1 t\right) & t_0 \le t \end{cases}$$



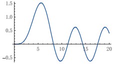
$$\begin{cases} p_0 \sin\left(\frac{\pi t}{t_0}\right) & 0 \le t \le t_0 \\ 0 & t_0 \le t \end{cases}$$



$$\begin{cases} \frac{p_0}{k}\frac{\pi\omega_1 t_0}{\omega_1^2 t_0^2 - \pi^2}\left(\frac{\omega_1 t_0}{\pi}\sin\frac{\pi t}{t_0} - \sin\omega_1 t\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\frac{\pi\omega_1 t_0}{\omega_1^2 t_0^2 - \pi^2}\left(-\sin\omega_1 (t - t_0) - \sin\omega_1 t\right) & t_0 \le t \end{cases}$$



$$\begin{cases} \frac{1}{2}p_0\left(1 - \cos\left(\frac{2\pi t}{t_0}\right)\right) & 0 \le t \le t_0 \\ 0 & t_0 \le t \end{cases}$$



$$\begin{cases} \frac{p_0}{2k}\left(1 - \frac{\omega_1^2 t_0^2}{\omega_1^2 t_0^2 - 4\pi^2}\left(\cos\left(\frac{2\pi t}{t_0}\right) - \frac{4\pi^2}{\omega_1^2 t_0^2}\cos\omega_1 t\right)\right) & 0 \le t \le t_0 \\ \frac{p_0}{k}\frac{2\pi^2}{\omega_1^2 t_0^2 - 4\pi^2}\left(\cos\omega_1 t - \cos\omega_1 (t - t_0)\right) & t_0 \le t \end{cases}$$

Exercises

1. With the help of Duhamel's tables, determine the response of a single-dof system to an exponentially decreasing loading. Double-check this result with numerical simulation
2. Compute the shock-response spectrum of a single-dof for other pulse shapes than those given p. 113.

Contents

Exploratory Exercise

1. Study the response of a single degree-of-freedom system to harmonic loading, i.e.

$$\ddot{q}(t) + 2\xi\omega_1\dot{q}(t) + \omega_1^2 q(t) = \sin(\Omega t)$$

starting from rest position, with $\omega_1 = 1$rad/s and $\xi = 0.05$. Compare the dynamic response $q(t)$ and the quasi-static response $q_s(t)$ for various values of Ω.
[nb: what if the loading frequency is much larger than the natural frequency? Much smaller? Similar?]

2. Represent, as a function of Ω, the amplitude of the steady-state response (i.e. after the transient phase).

Harmonic Loading I

Sinusoidal applied force

We thus have to solve

$$\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t) = \frac{\overline{p}}{m}\sin\overline{\omega}t$$

Calculus: second order differential equation, non-homogenous, with constant coefficients

The solution of such an equation is obtained by:

- writing *the* general solution of the homogenous equation

$$q_h(t) = e^{-\xi_1\omega_1 t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

- finding *one* particular solution $q_p(t)$
- writing the total solution $q = q_h + q_p$
- then constants A and B are determined from initial conditions

Harmonic Loading II

Study of three limit cases

$$\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t) = \frac{\overline{p}}{m}\sin\overline{\omega}t$$

Before application of the formal solution, we study three limit cases:

1. $\omega_1^2 q(t) \gg$
2. $\ddot{q}(t) \gg$
3. $2\omega_1\xi_1\dot{q}(t) \gg$

Harmonic Loading III

Case 1: $\omega_1^2 q(t) \gg (\ddot{q}(t), 2\omega_1 \xi_1 \dot{q}(t))$, i.e. $\frac{\overline{\omega}}{\omega_1} \ll$
The equation of motion simplifies to

$$\omega_1^2 q(t) = \frac{\overline{p}}{m} \sin \overline{\omega} t$$

whose solution is

$$q(t) = \frac{\overline{p}}{m\omega_1^2} \sin \overline{\omega} t = \frac{\overline{p}}{k} \sin \overline{\omega} t$$

Quasi-static response: the structure adapts at each time step to the applied force. Force and response are in phase.

Harmonic Loading IV

Case 2: $\ddot{q}(t) \gg \left(\omega_1^2 q(t), 2\omega_1\xi_1\dot{q}(t)\right)$, i.e. $\frac{\overline{\omega}}{\omega_1} \gg$

The equation of motion simplifies to

$$\ddot{q}(t) = \frac{\overline{p}}{m}\sin\overline{\omega}t$$

whose solution is

$$q = -\frac{\overline{p}}{m\overline{\omega}^2}\sin\overline{\omega}t + \text{cst} + \text{cst}.t \rightsquigarrow -\frac{\overline{p}}{k}\frac{1}{\frac{\overline{\omega}^2}{\omega_1^2}}\sin\overline{\omega}t$$

Inertial Response: governed by the structural mass. For $\overline{\omega} \gg$, the structure does not have enough time to adapt itself. Force and response are 180-degree out-of-phase.

Harmonic Loading V

Case 3: $2\omega_1\xi_1\dot{q}(t) \gg \left(\ddot{q}(t), \omega_1^2 q(t)\right)$, i.e. $\frac{\overline{\omega}}{\omega_1} \simeq 1$

nb. Rather uncommon because $\xi_1 \ll$. We study thus a very narrow band around ω_1.

We thus set $\overline{\omega} = \omega_1(1+\varepsilon)$, where $|\varepsilon| \ll 1$, so that

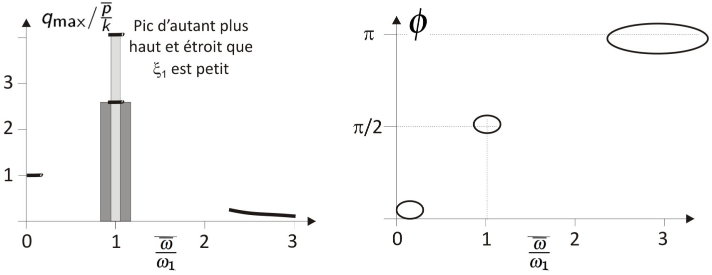
$$2\omega_1\xi_1\dot{q} = \frac{\overline{p}}{m}\sin\overline{\omega}t$$

whose solution is

$$q = \frac{-\overline{p}}{2m\omega_1\overline{\omega}\xi_1}\cos\overline{\omega}t + \text{cst} = \frac{-\cos\overline{\omega}t}{2(1+\varepsilon)\xi_1}\frac{\overline{p}}{k} + \text{cst}.$$

Resonance: the bandwidth is as narrow as ξ_1 is small, but dynamic amplification inversely proportional to ξ_1! Very dangerous phenomenon in structures.

Harmonic Loading VI



Behavior	$\frac{\overline{\omega}}{\omega_1}$	$q_{max}/\frac{\overline{p}}{k}$	ϕ
Quasi-static (k)	$\frac{\overline{\omega}}{\omega_1} \ll$	1	0
Resonant (c)	$\frac{\overline{\omega}}{\omega_1} \simeq 1$	$1/2\xi_1$	$\pi/2$
Inertial (m)	$\frac{\overline{\omega}}{\omega_1} \gg$	$1/\frac{\overline{\omega}^2}{\omega_1^2}$	π

Harmonic Loading VII

Sinusoidal applied force

We thus have to solve

$$\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t) = \frac{\overline{p}}{m}\sin\overline{\omega}t \qquad (27)$$

Calculus: second order differential equation, non-homogenous, with constant coefficients
The solution of such an equation is obtained by:

- writing *the* general solution of the homogenous equation

$$q_h(t) = e^{-\xi_1\omega_1 t}(A\cos\omega_d t + B\sin\omega_d t)$$

- finding *one* particular solution $q_p(t)$
- writing the total solution $q = q_h + q_p$
- then constants A and B are determined from initial conditions

Harmonic Loading VIII

Formal study of vibrations under harmonic loading

Observation: the homogenous solution q_{tr} fades away (if $\xi_1 \neq 0$). After a certain time (t_r), $q \simeq q_{st}$. We thus focus on q_{st} in the sequel.

q_{tr} is the transient component of the response;
q_{st} is the stationary component

We try to find a solution of the form

$$q_{st} = G_1 \sin \overline{\omega} t + G_2 \cos \overline{\omega} t. \tag{28}$$

Constants G_1 and G_2 are determined by substituting (28) in (27)

$$-\overline{\omega}^2 \left(G_1 \sin \overline{\omega} t + G_2 \cos \overline{\omega} t\right) + 2\overline{\omega}\omega_1\xi_1 \left(G_1 \cos \overline{\omega} t - G_2 \sin \overline{\omega} t\right) + \omega_1^2 \left(G_1 \sin \overline{\omega} t + G_2 \cos \overline{\omega} t\right) = \frac{\overline{p}}{m} \sin \overline{\omega} t.$$

Harmonic Loading IX

Balancing the coefficients of $\cos\overline{\omega}t$ and $\sin\overline{\omega}t$ provides

$$\begin{cases} \left(1-\beta^2\right) G_1 - 2\beta\xi_1 G_2 & = \frac{\overline{p}}{k} \\ 2\beta\xi_1 G_1 + \left(1-\beta^2\right) G_2 & = 0 \end{cases}$$

whose solution is

$$\begin{cases} G_1 & = \frac{\overline{p}}{k} \frac{1-\beta^2}{\left(1-\beta^2\right)^2+(2\xi_1\beta)^2} \\ G_2 & = \frac{\overline{p}}{k} \frac{-2\xi_1\beta}{\left(1-\beta^2\right)^2+(2\xi_1\beta)^2} \end{cases}$$

with $\beta = \frac{\overline{\omega}}{\omega_1}$

Alternative formulation of (28):

$$q_{st} = G_1 \sin\overline{\omega}t + G_2 \cos\overline{\omega}t = r\sin\left(\overline{\omega}t - \phi\right)$$

where

$$r = \frac{\overline{p}}{k} \frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\xi_1\beta\right)^2}}$$

$$\phi = \arctan\frac{2\xi_1\beta}{1-\beta^2}$$

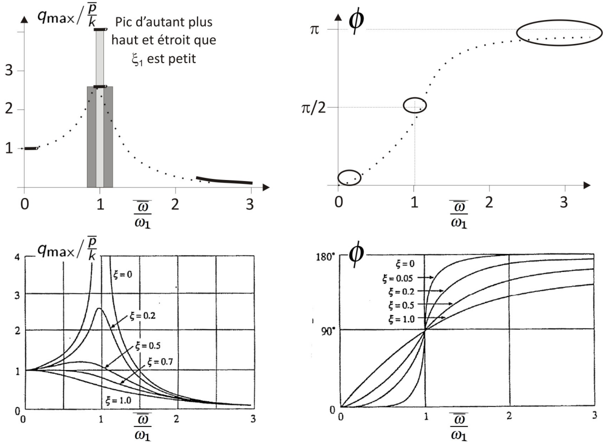
Harmonic Loading X

We may thus interpret:

- $\frac{q_{\max}}{\frac{\bar{p}}{k}} = \frac{1}{\sqrt{(1-\beta^2)^2+(2\xi_1\beta)^2}}$ as a dynamic amplification factor
- ϕ as a phase shift between the loading and the response
- $k\sqrt{(1-\beta^2)^2+(2\xi_1\beta)^2}$ as a dynamic stiffness, which connects the loading and the amplitude of the response

Harmonic Loading XI

We can now represent the exact expressions of $q_{max}/\frac{\overline{p}}{k}$ and ϕ , and compare them to the estimations given before.



Harmonic Loading XII

Complex Analysis

There is advantage in considering

$$\ddot{\mathcal{Q}} + 2\omega_1\xi_1\dot{\mathcal{Q}} + \omega_1^2\mathcal{Q} = \frac{\mathcal{P}}{m}e^{\imath\overline{\omega}t}. \tag{29}$$

rather than

$$\ddot{q} + 2\omega_1\xi_1\dot{q} + \omega_1^2 q = \frac{\overline{p}}{m}\sin\overline{\omega}t \tag{30}$$

[if $\mathcal{P}$ is real ($=\overline{p}$) and $\mathcal{Q}$ is the solution of (29), the solution q of (30) is given by $q = \Im(\mathcal{Q})$]

[but $\mathcal{P}$ may be complex...]

We search a solution of the form

$$Q_{st} = \mathcal{G}\,e^{\imath\overline{\omega}t}, \tag{31}$$

The substitution of (31) into (29) reads

$$-\overline{\omega}^2\mathcal{G}\,e^{\imath\overline{\omega}t} + 2\imath\omega_1\overline{\omega}\xi_1\mathcal{G}\,e^{\imath\overline{\omega}t} + \omega_1^2\mathcal{G}\,e^{\imath\overline{\omega}t} = \frac{\mathcal{P}}{m}e^{\imath\overline{\omega}t},$$

whose solution is (simply)

$$\mathcal{G} = \frac{\frac{\mathcal{P}}{m}}{-\overline{\omega}^2 + 2\imath\omega_1\overline{\omega}\xi_1 + \omega_1^2} = \mathcal{H}(\overline{\omega})\,\mathcal{P}, \tag{32}$$

Harmonic Loading XIII

where *the frequency response function (FRF)* $\mathscr{H}$ is defined by

$$\mathscr{H} = \frac{\frac{1}{k}}{1 - \frac{\overline{\omega}^2}{\omega_1^2} + 2\imath \frac{\overline{\omega}}{\omega_1}\xi_1} = \frac{\frac{1}{k}}{1 - \beta^2 + 2\imath\beta\xi_1}$$

With a harmonic load $\mathscr{P} e^{\imath\overline{\omega}t}$ is associated a harmonic response $Q_{st} = \mathscr{H}(\overline{\omega})\mathscr{P} e^{\imath\overline{\omega}t}$ with the same frequency $\overline{\omega}$, with an amplitude given by $|\mathscr{P}\mathscr{H}|$ and a phase shift corresponding to the phase of $\mathscr{P}\mathscr{H}$.

The frequency response function $\mathscr{H}$ may also be written

$$\mathscr{H} = \frac{1}{k} \frac{1 - \beta^2 - 2\imath\beta\xi_1}{\sqrt{(1-\beta^2)^2 + (2\beta\xi_1)^2}} = \frac{\frac{1}{k}}{\sqrt{(1-\beta^2)^2 + (2\beta\xi_1)^2}} e^{\imath \arctan \frac{2\xi_1\beta}{1-\beta^2}}$$

Harmonic Loading XIV

Transient study of the growth to resonance

When $\beta = 1$ ($\overline{\omega} = \omega_1$), the complete dynamic response reads

$$q = \underbrace{e^{-\xi_1 \omega_1 t} \left(A \cos \omega_d t + B \sin \omega_d t\right)}_{q_{tr}} \underbrace{- \frac{\overline{p}}{k} \frac{\cos \overline{\omega} t}{2\xi_1}}_{q_{st}}$$

With initial conditions at rest, one gets

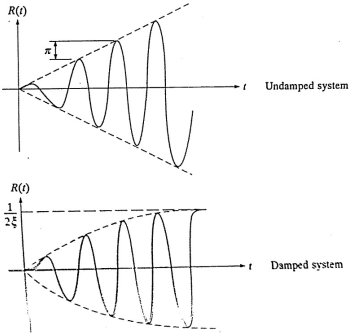
$$q = \frac{1}{2\xi_1} \frac{\overline{p}}{k} \left[e^{-\xi_1 \omega_1 t} \left(\cos \omega_d t + \frac{\xi_1}{\sqrt{1-\xi_1^2}} \sin \omega_d t \right) - \cos \omega_1 t \right].$$

or, for $\xi_1 \ll$:

$$q = \frac{1}{2\xi_1} \frac{\overline{p}}{k} \left[e^{-\xi_1 \omega_1 t} \xi_1 \sin \omega_1 t - \left(1 - e^{-\xi_1 \omega_1 t}\right) \cos \omega_1 t \right] \tag{33}$$

Harmonic Loading XV

Growth of the response when the frequency of the loading corresponds to one natural frequency

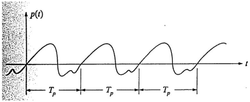


Harmonic Loading: Case Study I

Dolhain Footbridge / Kujawy Cement Mill

...

Periodic Loading I



Let us assume that the applied force $p(t)$ is periodic, with period T_0 (! not T_1)

The Fourier series decomposition of $p(t)$ reads

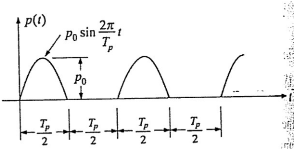
$$p = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t) \tag{34}$$

where $\Omega_0 = \frac{2\pi}{T_0}$ and coefficients a_n and b_n are given by

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} p(t)\,dt \\ a_n &= \frac{2}{T_0}\int_{-T_0/2}^{+T_0/2} p(t)\cos(n\Omega_0 t)\,dt \\ b_n &= \frac{2}{T_0}\int_{-T_0/2}^{+T_0/2} p(t)\sin(n\Omega_0 t)\,dt \end{aligned}$$

Periodic Loading II

Example: Runner-like loading



Coefficients of the Fourier series of $p(t)$ are given by:

$$a_0 = \frac{1}{T_0}\int_0^{+T_0/2} p_0 \sin\frac{2\pi t}{T_0}\,dt = \frac{p_0}{\pi}$$

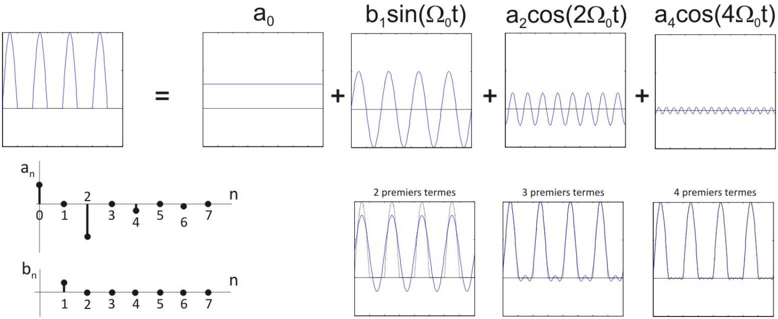
$$a_n = \frac{2}{T_0}\int_0^{+T_0/2} p_0 \sin\frac{2\pi t}{T_0}\cos\frac{2\pi n t}{T_0}\,dt = \begin{cases} 0 & \text{for } n \text{ odd} \\ \frac{p_0}{\pi}\frac{2}{1-n^2} & \text{for } n \text{ even} \end{cases}$$

$$b_n = \frac{2}{T_0}\int_0^{+T_0/2} p_0 \sin\frac{2\pi t}{T_0}\sin\frac{2\pi n t}{T_0}\,dt = \begin{cases} \frac{p_0}{2} & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}$$

Periodic Loading III

In (34), the summation $\sum_{n=1}^{\infty}$ may be truncated rapidly.

Proof (with example !?):



Periodic Loading IV

Superposition principle:

If q_1 is a solution of $\ddot{q}(t)+2\omega_1\xi_1\dot{q}(t)+\omega_1^2 q(t)=f_1$,
If q_2 is a solution of $\ddot{q}(t)+2\omega_1\xi_1\dot{q}(t)+\omega_1^2 q(t)=f_2$,
then $\alpha_1 q_1+\alpha_2 q_2$ is a solution of $\ddot{q}(t)+2\omega_1\xi_1\dot{q}(t)+\omega_1^2 q(t)=\alpha_1 f_1+\alpha_2 f_2$

Application to periodic loading ? Straightforward, because we can write

$$p=a_0+\sum_{n=1}^{\infty} a_n\cos(n\Omega_0 t)+\sum_{n=1}^{\infty} b_n\sin(n\Omega_0 t),$$

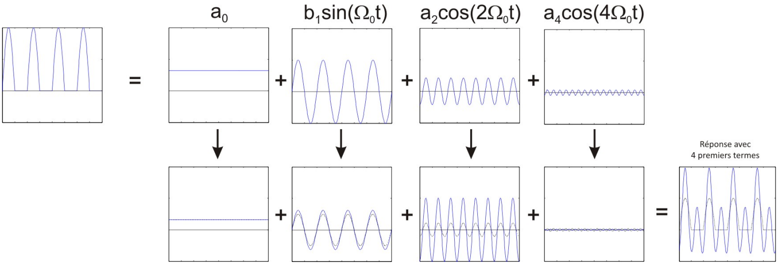
one just has to compute the response to each loading: a_0, $a_1\cos\Omega_0 t$, $a_2\cos 2\Omega_0 t$ ...,$b_1\sin\Omega_0 t$, $b_2\sin\Omega_0 t$, ...

Let q_0 the response under load a_0,
Let $q_{n,a}$ the response under load $a_n\cos(n\Omega_0 t)$,
Let $q_{n,b}$ the response under load $b_n\sin(n\Omega_0 t)$,

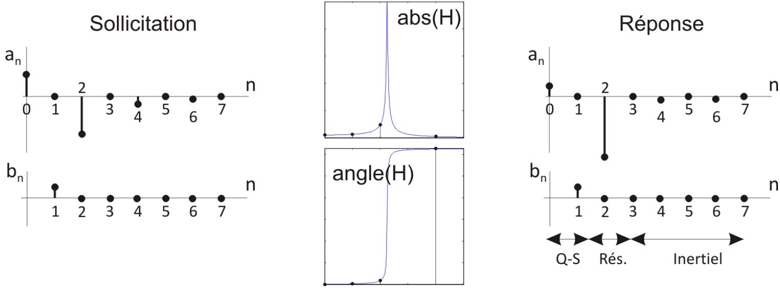
Periodic Loading V

The solution of the equation of motion reads

$$q(t) = q_0 + \sum_{n=1}^{\infty} q_{n,a}(t) + \sum_{n=1}^{\infty} q_{n,b}(t) \quad (35)$$



Periodic Loading VI



Periodic Loading VII

Alternative: decomposition as a complex Fourier series

The complex Fourier series decomposition of $p(t)$ reads

$$p = \sum_{n=-\infty}^{+\infty} \mathscr{P}_n e^{\imath n \Omega_0 t}$$

where the (complex) coefficients $\mathscr{P}_n$ are given by

$$\mathscr{P}_n = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} p(t)\, e^{-\imath n \Omega_0 t} dt \tag{36}$$

One then just has to study the response under each load $\mathscr{P}_0$, $\mathscr{P}_1 e^{\imath \Omega_0 t}$, $\mathscr{P}_2 e^{\imath 2 \Omega_0 t}$, etc.

Because, to a harmonic loading $\mathscr{P}_n e^{\imath n \Omega_0 t}$ corresponds a harmonic response

$$\mathscr{Q}_n = \mathscr{H}(n\Omega_0)\, \mathscr{P}_n e^{\imath n \Omega_0 t},$$

the complete response reads

$$\mathscr{Q} = \sum_{n=-\infty}^{+\infty} \mathscr{Q}_n = \sum_{n=-\infty}^{+\infty} \mathscr{H}(n\Omega_0)\, \mathscr{P}_n e^{\imath n \overline{\omega} t}.$$

This is the complex equivalent of (35).

Exercise: Periodic loading I

1. A man is jumping on a 5-m long beam. The generated force is assumed to be a half-sine function with amplitude $P_o = 1000N$

$$f(t) = P_o \sin\left(\frac{2\pi t}{T_0}\right) \rightarrow P(t) = \begin{cases} f(t) & \text{if } f(t) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

and adjustable period of loading T_0. The beam is assumed to have a bending stiffness $EI = 4 \cdot 10^4 N.m^2$, a mass per unit length $\mu = 50kg/m$ and a damping ratio $\xi = 0.05$. Compute the maximum displacement under this dynamic loading, if we assume that the man has tuned his jumping frequency to the worst case. [nb: the result may be computed with hand calculations]

2. Validate your findings with a step-by-step simulation.

Exercise: Periodic loading II

"A" worst case is when $\Omega_0 = \omega_1$, i.e. $T_0 = T_1$. In that case, only the b_1 component of the loading matters. The response to $P_0/2 \cdot \sin(\Omega_0 t) = P_0/2 \cdot \sin\left(\frac{2\pi}{T_1} t\right)$ is

$$q_{b_1} = \frac{P_0}{2} \frac{1}{2\xi k} = 0.32\,m$$

(add qs component $P_0/k = 0.064m$)

Another worst case is when $2\Omega_0 = \omega_1$, i.e. $\frac{T_0}{2} = T_1$. In that case, only the a_1 component of the loading matters. The response to $-\frac{2}{3\pi} P_0 \cdot \sin(2\Omega_0 t) = -\frac{2}{3\pi} P_0 \cdot \sin\left(\frac{2\pi}{T_1} t\right)$ is

$$q_{a_2} = \frac{2}{3\pi} P_0 \frac{1}{2\xi k} = 0.14\,m$$

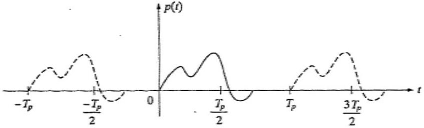
(add qs component $P_0/k = 0.064m$, and qs component for $b_1 - term$, $P_0/2/k = 0.032$).

Exercise: Periodic loading III

\movie[width=8cm,height=6cm,poster,borderwidth=1pt,showcontrols

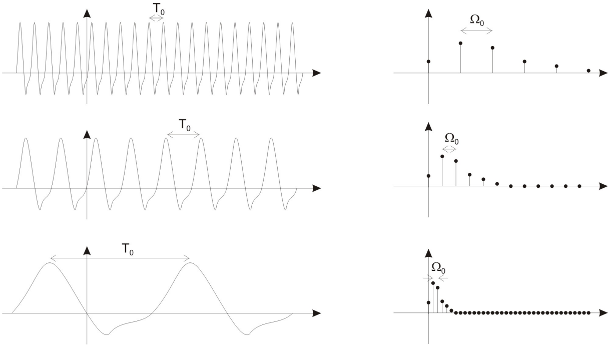
Arbitrary loading I

An arbitray loading is just a periodic loading with infinite period



$$p(t) = \lim_{T_0 \to \infty} \sum_{n=-\infty}^{+\infty} \mathscr{P}_n e^{\imath n \Omega_0 t} \tag{37}$$

Arbitrary loading II



- Because $T_0\Omega_0 = 2\pi$, the limit $T_0 \to \infty$ implies $\Omega_0 \to 0$.
- Ω_0 is the frequency resolution
- the limit means therefore that $\mathscr{P}_n$ becomes a continuous fonction of frequency $\mathscr{P}(\Omega)$.

Arbitrary loading III

$$p(t) = \lim_{T_0 \to \infty} \sum_{n=-\infty}^{+\infty} \mathscr{P}_n e^{\imath n \Omega_0 t}$$

The limit of a summation must give an integral → need to introduce a small frequency step $\Delta\Omega$ ($\equiv \Omega_0$):

$$p(t) = \lim_{T_0 \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} T_0 \mathscr{P}_n e^{\imath \Omega_n t} \Delta\Omega$$

$$= \lim_{T_0 \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \mathscr{P}(\Omega_n) e^{\imath \Omega_n t} \Delta\Omega$$

where we have set $\mathscr{P}(\Omega_n) = T_0 \mathscr{P}_n$. Hence:

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathscr{P}(\Omega) e^{\imath \Omega t} d\Omega$$

(definition of the inverse Fourier transform of $\mathscr{P}(\Omega)$) .

Arbitrary loading IV

The limit $T_0 \to +\infty$ applied to $\mathscr{P}(\Omega_n) = T_0 \mathscr{P}_n$ gives, by considering (36):

$$\mathscr{P}(\Omega) = \lim_{T_0 \to \infty} T_0 \mathscr{P}_n = \lim_{T_0 \to \infty} \int_{-T_0/2}^{+T_0/2} p(t)\, e^{-\imath \Omega_n t} dt$$

i.e.

$$\mathscr{P}(\Omega) = \int_{-\infty}^{+\infty} p(t)\, e^{-\imath \Omega t} dt. \tag{38}$$

Summary

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathscr{P}(\Omega)\, e^{\imath \Omega t} d\Omega \tag{39}$$

$$\mathscr{P}(\Omega) = \int_{-\infty}^{+\infty} p(t)\, e^{-\imath \Omega t} dt. \tag{40}$$

Relations (39) and (40) indicate that functions $p(t)$ and $\mathscr{P}(\Omega)$ are Fourier conjugates

Arbitrary loading V

The Fourier series is applicable to the loading as well as to the response

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathscr{P}(\Omega)\, e^{\imath\Omega t} d\Omega \quad \longleftrightarrow \quad \mathscr{P}(\Omega) = \int_{-\infty}^{+\infty} p(t)\, e^{-\imath\Omega t} dt.$$

$$q(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathscr{Q}(\Omega)\, e^{\imath\Omega t} d\Omega \quad \longleftrightarrow \quad \mathscr{Q}(\Omega) = \int_{-\infty}^{+\infty} q(t)\, e^{-\imath\Omega t} dt. \tag{41}$$

Dynamic Analysis

Let a infinitesimal bandwidth $\frac{d\Omega}{2\pi}$. The corresponding loading is $\mathscr{P}(\Omega)\, e^{\imath\Omega t} \frac{d\Omega}{2\pi}$. It provides an elementary contribution to the reponse $\mathscr{H}(\Omega)\, \mathscr{P}(\Omega)\, e^{\imath\Omega t} \frac{d\Omega}{2\pi}$. One obtains the dynamic response by considering the whole frequency domain

$$q(t) = \int_{-\infty}^{+\infty} \mathscr{H}(\Omega)\, \mathscr{P}(\Omega)\, e^{\imath\Omega t} \frac{d\Omega}{2\pi}.$$

$$\rightarrow \mathscr{Q}(\Omega) = \mathscr{H}(\Omega)\, \mathscr{P}(\Omega)$$

An Impressive Shortcut...

We multiply the equation of motion by $e^{-\imath\Omega t}$ (side-by-side) and integrate along time:

$$\int_{-\infty}^{+\infty} \left(\ddot{q} + 2\omega_1\xi_1\dot{q} + \omega_1^2 q\right) e^{-\imath\Omega t} dt = \int_{-\infty}^{+\infty} \frac{p(t)}{m} e^{-\imath\Omega t} dt. \quad (42)$$

This swaps the parameters (t to Ω):

$$-\Omega^2\mathcal{Q}(\Omega) + 2\imath\omega_1\xi_1\Omega\mathcal{Q}(\Omega) + \omega_1^2\mathcal{Q}(\Omega) = \frac{\mathcal{P}(\Omega)}{m}.$$

Because $\mathcal{H}(\Omega) = \frac{\omega_1^2}{k}\left(-\Omega^2 + 2\imath\omega_1\xi_1\Omega + \omega_1^2\right)^{-1}$, we thus have

$$\mathcal{Q}(\Omega) = \mathcal{H}(\Omega)\mathcal{P}(\Omega)$$

(much more simple, but without a simple physical intuition ...)

---

nb: by integrating successively by part, it is possible to show that

$$\int_{-\infty}^{+\infty} \ddot{q}e^{-\imath\Omega t} dt = -(-\imath\Omega)^2 \int_{-\infty}^{+\infty} qe^{-\imath\Omega t} dt = -\Omega^2\mathcal{Q}(\Omega), \quad \int_{-\infty}^{+\infty} \dot{q}e^{-\imath\Omega t} dt = -(-\imath\Omega)\int_{-\infty}^{+\infty} qe^{-\imath\Omega t} dt = \imath\Omega\mathcal{Q}(\Omega)$$

Exercises With Matlab

1. Compute the Fourier transform of $p(t) = \frac{1}{1+t^2}$.
   [$\mathscr{P}(\omega) = \sqrt{\frac{\pi}{2}} e^{-|\omega|}$]

2. Compare the analytical result obtained in #1, to the numerical estimation in Matlab.

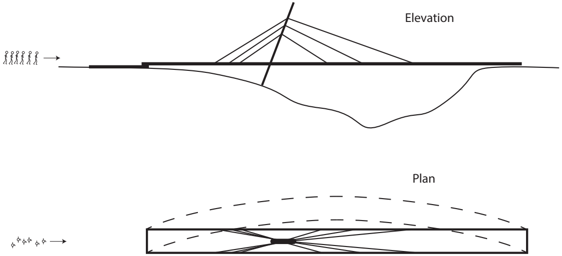
3. Compute the response of a single degree-of-freedom system to the loading $p(t)$. Compare both time and frequency domain solutions [choose $m = 1kg$, $\omega_1 = 1rad/s$ and $\xi = 0.05$].

Exercises I

1. A control console containing delicate instrumentation is to be located on the floor of a test laboratory where it has been determined that the floor slab is vibrating vertically with an amplitude of 0,8 mm at 20 Hz. If the mass of the console is 363 kg, determine the stiffness of the vibration isolation system required to reduce the vertical motion amplitude of the console to 0,013 cm.
2. ***[Juin 2016]*** Des études ont montré que les piétons marchant sur une passerelle flexible sont susceptible de synchroniser leur marche lorsque l'accélération horizontale du tablier dépasse 0.1m/s^2. Une foule de 0.5 personne par mètre carré, avançant lentement, à une vitesse de 0.7m/s, se prépare à traverser une passerelle de 100m de portée et de 5m de large, réalisée en acier soudé. Le premier mode propre de la passerelle dans le plan horizontal peut être approché par une forme sinusoïdale à une demi-onde, de masse généralisée 100 tonnes et fréquence propre $f_1 = 0.45$Hz. Sachant que la sollicitation horizontale générée par un piéton est modélisée comme une action harmonique d'intensité $F_0 = 20N$ et que la

Exercises II

fréquence de la sollicitation horizontale d'une marche lente se trouve entre 0.4 Hz et 0.9 Hz, déterminez **(i)** s'il y lieu de craindre une synchronisation de la foule, **(ii)** le temps nécessaire pour atteindre le régime stationnaire, c'est-à-dire l'accélération maximale de la passerelle dans son premier mode.



NB: quand une foule de N personnes déambule librement, tous les piétons ne peuvent pas synchroniser leur marche. On admet donc généralement que l'action de la foule peut être représentée par un ensemble, plus petit, de $\sqrt{N}$ personnes parfaitement synchronisées entre elles et avec la passerelle, tout en négligeant les contributions des autres piétons.

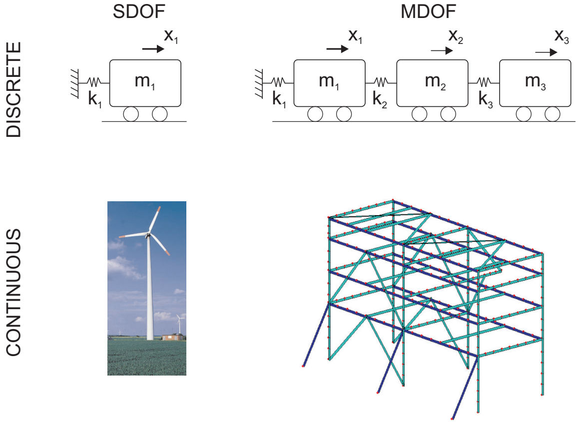
Section IV: Multi Degree-Of-Freedom Systems

Learning Outcomes:

- dynamical modeling of multi degree-of-freedom structures
- existence and computation of eigen modes and frequencies
- simple and hand calculation of natural frequencies
- time and frequency domain analysis of large structure

Contents

MDOF Systems

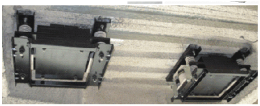


▷ either the structure is a discrete set of rigid bodies connected to each other

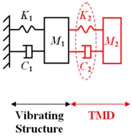
▷ either the structure is continuous ($\infty-$number of DOFs) and has to be discretized to be studied numerically

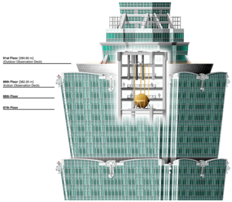
Discrete Systems I

Example of discrete systems in civil engineering applications:
Tuned Mass Dampers

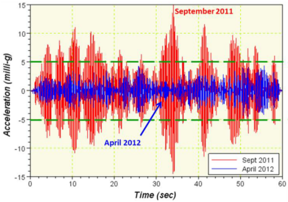


Floor dampers (source: deicon.com)



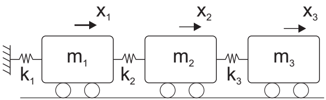


TAIPEI (source: Wikipedia)



Discrete Systems II

Setting up the equation of motion



$$\begin{cases} -m_1\ddot{x}_1 - k_1 x_1 + k_2\left(x_2 - x_1\right) + p_1 = 0 \\ -m_2\ddot{x}_2 + k_3\left(x_3 - x_2\right) + k_2\left(x_1 - x_2\right) + p_2 = 0 \\ -m_3\ddot{x}_3 + k_3\left(x_2 - x_3\right) + p_3 = 0 \end{cases}$$

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

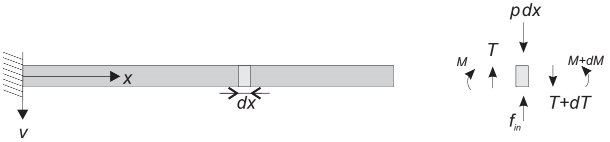
More generally:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t)$$

Continuous Systems I

Continuous Version

Example: Transverse vibrations of a Bernoulli beam



$$\mu \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v(x,t)}{\partial x^2}\right) = p(x,t)$$

with μ the lineic mass (mass per unit length) and EI the bending stiffness.

$$T = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI \frac{\partial^2 v}{\partial x^2}\right) \quad \text{and} \quad f_i = \mu\, dx \frac{\partial^2 v}{\partial t^2}$$

[Beam equation, with inertial forces]

How to get rid of the PDE ?

Lumped modeling → 1-DOF

1. assume the response takes place in a given shape $v(x,t) = \phi(x)\, q(t)$ ($\phi(x)$ satisfies boundary conditions)
2. project the response in the assumed shape

$$m^\star \ddot{q}(t) + k^\star q(t)\, dx = \int_\Omega \phi(x)\, p(x,t)\, dx$$

with $m^\star = \int_\Omega \mu \phi^2(x)\, dx$ and $k^\star = \int_\Omega \phi(x) \frac{d^2}{dx^2}\left(EI \frac{d^2 v(x,t)}{dx^2}\right) dx$.

[nb: Another way to cope with the difficulty of the PDE is to recourse to eigen functions - See section related to vibrations of continuous systems]

[nb: Another alternative is the finite element method (transformation of an ODE into a set of algebraic equations - See Discrete Version]

Continuous Systems III

How to get rid of the PDE ?

Finite Element (Displacement/Rotation) Method → M-DOF

Static Analysis (Finite Element Method):

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v(x,t)}{\partial x^2}\right)=p(x,t) \quad \rightarrow \quad \mathbf{K}\mathbf{x}=\mathbf{p}$$

Dynamic Analysis:

$$\mu\frac{\partial^2 v(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v(x,t)}{\partial x^2}\right)=p(x,t) \quad \rightarrow \quad \mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{p}(t)$$

- **M**: mass → inertial forces
- **K**: stiffness → internal forces

Continuous Systems IV

With the method of displacements (or more generally the finite element method) the equilibrium of a structure reads

$$\mathbf{K}\,\mathbf{x} = \mathbf{p}$$

where

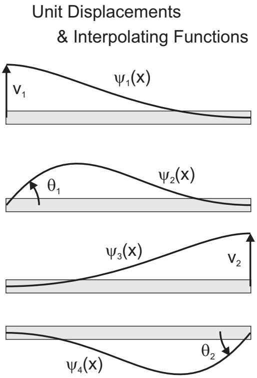
- K_{ij} represents the reaction at DOF i under a unit displacement at DOF j, while all other DOFs remain blocked;
- p_i represents the energically equivalent force applied at DOF i

$$K_{ij} = \int \psi_i''(x)\, EI(x)\, \psi_j''(x)\, dx$$
$$p_i = \int \psi_i(x)\, p(x)\, dx$$

with $\psi_i(x)$ the interpolation function (Hermite polynomials)

Continuous Systems V

Example: bending deformation of a beam



$$\psi_1 = 1 - 3\left(\frac{x}{\ell}\right)^2 + 2\left(\frac{x}{\ell}\right)^3$$

$$\psi_2 = \ell\left[\frac{x}{\ell} - 2\left(\frac{x}{\ell}\right)^2 + \left(\frac{x}{\ell}\right)^3\right]$$

$$\psi_3 = 3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3$$

$$\psi_4 = \ell\left[-\left(\frac{x}{\ell}\right)^2 + \left(\frac{x}{\ell}\right)^3\right]$$

Continuous Systems VI

This results in $\left(K_{ij} = \int \psi_i''(x)\, EI(x)\, \psi_j''(x)\, dx\right)$

$$\mathbf{K}_e = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}$$

For one finite element

$$\mathbf{K}_e \mathbf{x}_e = \mathbf{p}_e$$

with $\mathbf{x}_e = \left\langle \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \right\rangle^T$

$\rightarrow$ internal forces (LHS) = external applied forces (RHS).

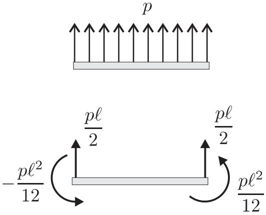
[nb: use this equation to internal forces (LHS) - in a postprocessing operation]

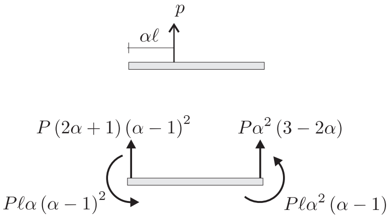
Element stiffness matrices $\mathbf{K}_e$ and the vector of applied loads $\mathbf{p}_e$ are then localized, rotated and assembled in order to obtain the assembled system

$$\mathbf{K}\,\mathbf{x} = \mathbf{p}$$

Continuous Systems VII

Example: vector of work-equivalent applied forces ($p_i = \int \psi_i(x)\, p(x)\, dx$)





Post-treatment (if required)

- the displacement field along a finite element is obtained by interpolation

$$v(x) = v_1\psi_1(x) + \theta_1\psi_2(x) + v_2\psi_3(x) + \theta_2\psi_4(x)$$

- nodal internal forces of a finite element are obtained by $\mathbf{K}_e \mathbf{x}_e$

Continuous Systems VIII

Mass Matrix

$\mathbf{Kx} = \mathbf{p}$ translates the equilibrium of the nodes of the model. In dynamics, we have to add inertial forces.

Option 1: Consistent mass matrix

Let $v(x,t)$ the deformed configuration of the finite element. The inertial forces (per unit length) are given by $f_I(x) = \mu\, \ddot{v}(x,t)$.

Work-equivalent inertial forces (corresponding to DOF i) are given by

$$\begin{aligned} F_{I,i}(t) &= \int_0^\ell f_I(x,t)\psi_i(x)dx \\ &= \int_0^\ell \mu \left[\ddot{v}_1\psi_1(x) + \ddot{\theta}_1\psi_2(x) + \ddot{v}_2\psi_3(x) + \ddot{\theta}_2\psi_4(x)\right]\psi_i(x)\,dx \\ &= \mu \int_0^\ell \psi_i(x) \left\langle \psi_1(x) \quad \psi_2(x) \quad \psi_3(x) \quad \psi_4(x) \right\rangle \ddot{\mathbf{x}}_e\, dx \end{aligned}$$

with $\ddot{\mathbf{x}}_e = \left\langle \ddot{v}_1 \quad \ddot{\theta}_1 \quad \ddot{v}_2 \quad \ddot{\theta}_2 \right\rangle^T$

Continuous Systems IX

For one finite element

$$\mathbf{M}_e \ddot{\mathbf{x}}_e = \mathbf{F}_{I,e}$$

where $M_{e_{ij}} = \int \mu \psi_i(x) \psi_j(x)\, dx$, i.e.

$$\mathbf{M}_e = \frac{\mu \ell}{420} \begin{pmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{pmatrix}$$

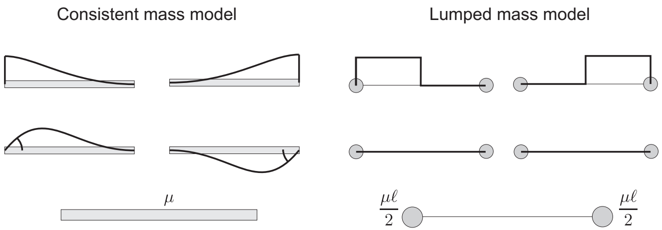
Element mass matrices $\mathbf{M}_e$ are then localized, rotated and assembled in order to obtain the assembled matrix $\mathbf{M}$ and the discrete version of the equation of motion

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}$$

$$\mu \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v(x,t)}{\partial x^2} \right) = p(x,t) \quad \rightarrow \quad \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t)$$

done!

Continuous Systems X



Interest in considering simpler interpolation functions ψ_i ???

Continuous Systems XI

Option 2: Lumped mass matrix

The force at node i is given by the mass at that node multiplied by its acceleration. We thus define

$$\mathbf{M}_e = \begin{pmatrix} \frac{\mu\ell}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu\ell}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that $\mathbf{M}_e\ddot{\mathbf{x}}_e$ corresponds to the nodal energetically equivalent inertial forces along the element.

Continuous Systems XII

The damping matrix could be established in a similar manner...
...provided the evolution of viscosity along the finite element is known $c(x)$.

$$F_{D,i}(t) = \int f_D(x,t)\psi_i(x)dx$$

$$= \int c(x)\left[\dot{v}_1\psi_1(x) + \dot{v}_2\psi_2(x) + \dot{\theta}_1\psi_3(x) + \dot{\theta}_2\psi_4(x)\right]\psi_i(x)\,dx$$

$$= \int_0^\ell c(x)\,\psi_i(x)\left\langle \psi_1(x) \quad \psi_2(x) \quad \psi_3(x) \quad \psi_4(x) \right\rangle \dot{\mathbf{x}}_e\,dx$$

with $\dot{\mathbf{x}}_e = \left\langle \dot{v}_1 \quad \dot{\theta}_1 \quad \dot{v}_2 \quad \dot{\theta}_2 \right\rangle^T$

For one finite element

$$\mathbf{C}_e\dot{\mathbf{x}}_e = \mathbf{F}_{D,e}$$

where $C_{e_{ij}} = \int c(x)\psi_i(x)\psi_j(x)\,dx$.

Continuous Systems XIII

Damping in structures in composed of

- internal/material damping → few information (use material specific damping ratio)
- additional damping (shock absorbers, dash-pots, dampers) → model required
- aerodynamic damping→ model required

Dash-pot model (with constant viscosity c, i.e. `Force = Viscosity` × `Relative Velocity`),

$$\begin{cases} \psi_1 = 1 - \frac{x}{\ell} \\ \psi_2 = \frac{x}{\ell} \end{cases} \rightarrow \quad \mathbf{C}_e = \begin{pmatrix} c & -c \\ -c & c \end{pmatrix}$$

nb: same as $\mathbf{K}_e = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$ for a spring/bar model with constant stiffness k, i.e. `Force = Stiffness` × `Relative Displacement`

Continuous Systems XIV

In summary:

- Compute element matrices $\mathbf{K}_e$ and $\mathbf{M}_e$, rotate then assemble $\rightarrow$ $\mathbf{K}$ and $\mathbf{M}$

- Construct structural damping $\rightarrow$ $\mathbf{C}_s$
[usually as a linear combination of $\mathbf{K}$ and $\mathbf{M}$, the only reliable information... see next]

- Compute additional element matrices corresponding to dampers $\mathbf{C}_d$ and to aerodynamic damping $\mathbf{C}_a$

Finally

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{p}(t)$$

with $\mathbf{C}=\mathbf{C}_s+\mathbf{C}_d+\mathbf{C}_a$.

Numerical Solution of the MDOF equation

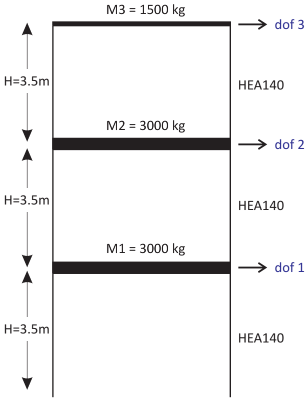
1-DOF system:

$$\left(\frac{1}{\alpha\Delta t^2}m+\frac{\delta}{\alpha\Delta t}c+k\right)q_{t+\Delta t}=p_{t+\Delta t}+c\left(\frac{\delta}{\alpha\Delta t}q_t+\left(\frac{\delta}{\alpha}-1\right)\dot{q}_t+\frac{\Delta t}{2}\left(\frac{\delta}{\alpha}-2\right)\ddot{q}_t\right)$$
$$+m\left(\frac{1}{\alpha\Delta t^2}q_t+\frac{1}{\alpha\Delta t}\dot{q}_t+\left(\frac{1}{2\alpha}-1\right)\ddot{q}_t\right)$$

M-DOF system:

$$\left(\frac{1}{\alpha\Delta t^2}\mathbf{M}+\frac{\delta}{\alpha\Delta t}\mathbf{C}+\mathbf{K}\right)\mathbf{x}_{t+\Delta t}=\mathbf{p}_{t+\Delta t}+\mathbf{C}\left(\frac{\delta}{\alpha\Delta t}\mathbf{x}_t+\left(\frac{\delta}{\alpha}-1\right)\dot{\mathbf{x}}_t+\frac{\Delta t}{2}\left(\frac{\delta}{\alpha}-2\right)\ddot{\mathbf{x}}_t\right)$$
$$+\mathbf{M}\left(\frac{1}{\alpha\Delta t^2}\mathbf{x}_t+\frac{1}{\alpha\Delta t}\dot{\mathbf{x}}_t+\left(\frac{1}{2\alpha}-1\right)\ddot{\mathbf{x}}_t\right)$$

Exploratory Exercises I



$$\mathbf{M} = \begin{bmatrix} 3000 & & \\ & 3000 & \\ & & 1500 \end{bmatrix} \mathrm{kg}$$

$$\mathbf{K} = \begin{bmatrix} 2.43 & -1.21 & 0 \\ -1.21 & 2.43 & -1.21 \\ 0 & -1.21 & 1.21 \end{bmatrix} .10^6 \mathrm{N/m}$$

$$\mathbf{C} = \begin{bmatrix} 3000 & & \\ & 3000 & \\ & & 1500 \end{bmatrix} \mathrm{Ns/m}$$

Determine the response of a 3-storey building subjected to a support motion $u_g = A\sin\left[\Omega(t)\,t\right]$, with $\Omega(t) = \Omega_0 + \upsilon t$, ($A = 0.01$, $\Omega_0 = 1\mathrm{rad/s}$, $\upsilon = 0.1\mathrm{rad/s}$ and $t \in [0;500]\,s$)

Analytical Study of the equation of motion I

Newmark is fine, but does not provide a clear understanding of what is going on...

SDOF

- Free response
- Impulsive Loading $\rightarrow$ arbitrary loading $\equiv$ sequence of pulses
- Harmonic Loading $\rightarrow$ arbitrary loading $\equiv$ sum of harmonic loadings (Fourier series .vs. Fourier transform)

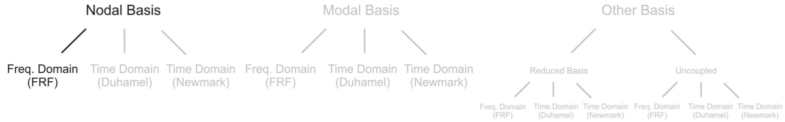
MDOF

Same outline, but in a nodal basis or in a modal basis

- Harmonic Loading $\rightarrow$ arbitrary loading $\equiv$ sum of harmonic loadings
- Impulsive Loading $\rightarrow$ arbitrary loading $\equiv$ sequence of pulses
- Free response

Contents

Frequency Domain I



Frequency Domain II

Apply a side-by-side Fourier transform to both sides of the equation of motion

$$\mathbf{M}\left(-\omega^2\mathscr{X}(\omega)\right)+\mathbf{C}\left(\imath\omega\mathscr{X}(\omega)\right)+\mathbf{K}\mathscr{X}(\omega)=\mathscr{P}(\omega)$$
$$\Rightarrow\left(-\mathbf{M}\omega^2+\imath\omega\mathbf{C}+\mathbf{K}\right)\mathscr{X}(\omega)=\mathscr{P}(\omega)$$

or

$$\mathscr{X}(\omega)=\mathscr{H}(\omega)\mathscr{P}(\omega)$$

where $\mathscr{H}=\left(-\mathbf{M}\omega^2+\imath\omega\mathbf{C}+\mathbf{K}\right)^{-1}$ is the **M-DOF Frequency Response Function**

We can also write

$$\mathscr{X}_i(\omega)=\sum_{j=1}^{N}\mathscr{H}_{ij}(\omega)\mathscr{P}_j(\omega)$$

Frequency Domain III

Assume momentarily that $\mathscr{P}_j(\omega) = 0, \forall j \in [0; N] \setminus k$ (only one force is applied to the structure). In that case:

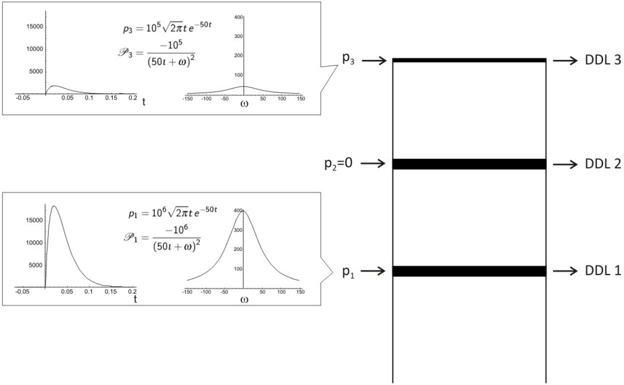
$$\mathscr{X}_i(\omega) = \mathscr{H}_{ik}(\omega)\,\mathscr{P}_k(\omega)$$

- $\mathscr{H}_{ik}(\omega)$ is the (complex) response of DOF i when a unit harmoinc loading is applied at DOF j
- $|\mathscr{H}_{ik}|$ represents the amplitude of the response and $\angle\mathscr{H}_{ik}$ is the phase shift between the loading and the response.

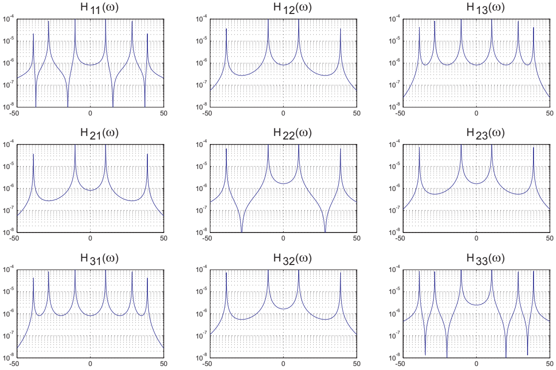
→ same meaning as in the S-DOF case

Frequency Domain IV

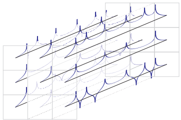
Example



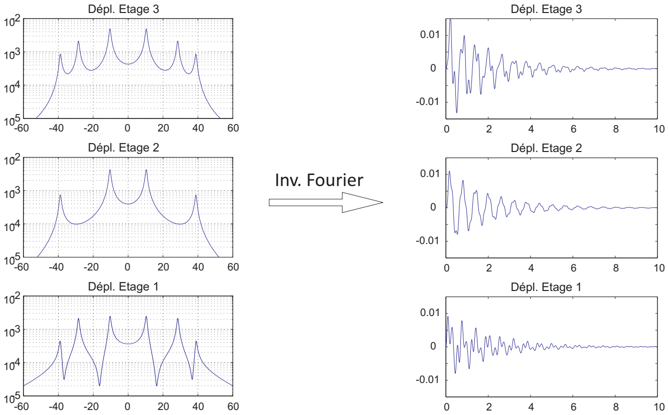
Frequency Domain V



En pratique: une matrice 3 x 3 x N_{freq}



Frequency Domain VI



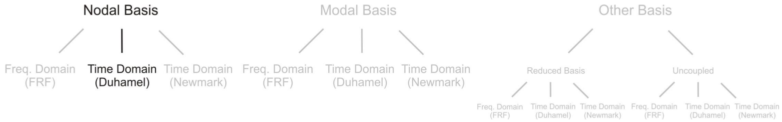
Frequency Domain VII

Vibrations of a building equipped with a antenna on its roof

Méca II, 2017-2018

V. Denoël

Time Domain Analysis I



Time Domain Analysis II

Integration of the equation of motion along a very short time window gives

$$\int_0^{\Delta t} \left(\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t)\right) dt = \int_0^{\Delta t} \mathbf{p}(t)\, dt$$

... or (after some developments):
the response is a free response with initial conditions,

$$\mathbf{x}(0) = 0$$

$$\dot{\mathbf{x}}(0^+) = \mathbf{M}^{-1}\mathbf{I}$$

By extension of the unit impulsive response S-DOF:

The unit impulsive response $\mathbf{h}_{i,j}$ is the response of DOF i when a unit impulsive force is applied at DOF k.

Time Domain Analysis III

SDOF: $q(t) = \int_0^{+\infty} h(t-\tau)\, p(\tau)\, d\tau$

Step 1. Consider the arbitrary force applied at DOF j. This force is decomposed as a sequences of pulses → Duhamel's convolution integral:

$$x_i(t) = \int_0^{+\infty} h_{i,j}(t-\tau)\, p_j(\tau)\, d\tau$$

Step 2. Repeat the operation for forces applied at other DOFs

$$x_i(t) = \sum_{j=1}^{N} \int_0^{+\infty} h_{i,j}(t-\tau)\, p_j(\tau)\, d\tau$$

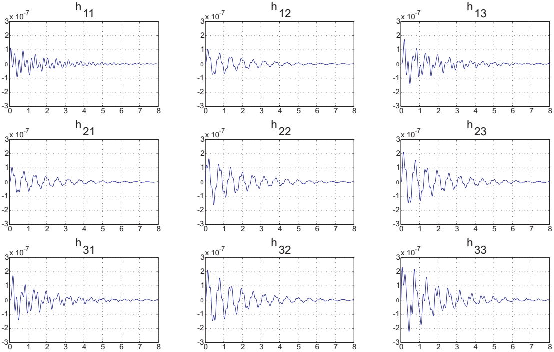
$$= \int_0^{+\infty} \sum_{j=1}^{N} h_{i,j}(t-\tau)\, p_j(\tau)\, d\tau$$

or, with a matrix format

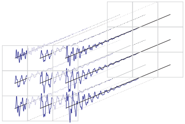
$$\mathbf{x}(t) = \int_0^{+\infty} \mathbf{h}(t-\tau)\, \mathbf{p}(\tau)\, d\tau$$

Time Domain Analysis IV

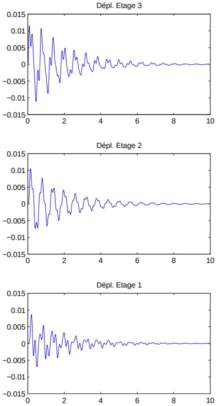
Example



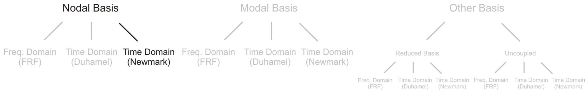
En pratique: une matrice 3 x 3 x N_{step}



Time Domain Analysis V



Nodal Step-by-step Analysis



Méca II, 2017-2018

V. Denoël

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Modal Basis Analysis

Méca II, 2017-2018

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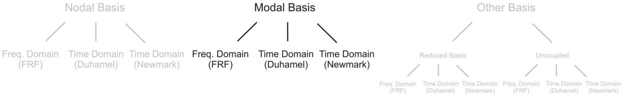


Undamped Free Vibrations I

We study *undamped vibrations*

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0} \tag{43}$$

... and try a solution of the form (separation of variables)

$$\mathbf{x}(t) = \phi\, q(t) \tag{44}$$

where $q(t)$ is a **scalar** function and ϕ is a vector
→ all nodes are moving in phase; the ratio of two amplitudes is always constant $(\forall t)$.
→ ϕ is a shape (as a static deformation)

Substitution of (44) in(43) gives

$$\mathbf{M}\phi\ddot{q}(t) + \mathbf{K}\phi q(t) = \mathbf{0} \tag{45}$$

$\mathbf{M}\phi$ and $\mathbf{K}\phi$ are vectors $(N \times 1)$. Let

$$\mathbf{m} = \mathbf{M}\phi \quad ; \quad \mathbf{k} = \mathbf{K}\phi.$$

Undamped Free Vibrations II

Equation (45) is written component-by-component:

$$m_i \ddot{q}(t) + k_i q(t) = 0 \quad \Rightarrow \quad \frac{k_i}{m_i} = -\frac{\ddot{q}(t)}{q(t)}$$

Left $fct(i)$, right $fct(t) \Rightarrow$ both members are equal to a positive constant (independent from i and t)

Hence

$$\frac{k_i}{m_i} = -\frac{\ddot{q}(t)}{q(t)} = \omega^2$$

where ω^2 is a constant (not any!). And so

$$\begin{cases} \ddot{q}(t) + \omega^2 q(t) = 0 \\ k_i = m_i \omega^2 \end{cases} \quad \Rightarrow \quad \text{harmonic response}$$

This indicates that ω has the physical meaning of a circular frequency.

Undamped Free Vibrations III

One still has to determine the value(s) of ω and the shape ϕ...

$$k_i = m_i \omega^2 \quad \Rightarrow \quad \mathbf{K}\phi = \mathbf{M}\phi\omega^2 \quad \Leftrightarrow \quad \left(\mathbf{K} - \mathbf{M}\omega^2\right)\phi = \mathbf{0} \tag{46}$$

From calculs... we know:

- ω^2 is the eigen value of the general problem with $\mathbf{K}$ et $\mathbf{M}$, and ϕ is the corresponding eigen vector $\rightarrow (\omega_i, \phi_i)$
- there exist N pairs (ω_i, ϕ_i) satisfying (46)

In practice (analytical approach)

The eigen values ω_i^2 are determined from

$$\det\left(\mathbf{K} - \mathbf{M}\omega^2\right) = 0$$

$\rightarrow$ degree-N polynomial in ω^2 with N roots (positive because $\mathbf{K}$ and $\mathbf{M}$ are positive definite)

$$0 < \omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_N^2$$

Undamped Free Vibrations IV

Then, for a given ω_i, the corresponding eigen vector ϕ_i is obtained by solving

$$\left(\mathbf{K} - \mathbf{M}\omega_i^2\right)\phi_i = \mathbf{0}$$

The system is singular (because $\det\left(\mathbf{K} - \mathbf{M}\omega_i^2\right) = 0$), so ϕ_i is defined with a **multiplicative constant**.

To simplify notations, the eigen vectors are gathered in a matrix

$$\mathbf{\Phi} = [\phi_1 \quad \phi_2 \quad \dots \quad \phi_N]$$

and the eigen values in a diagonal matrix

$$\Omega^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_N^2 \end{bmatrix}$$

so that

$$\left(\mathbf{K} - \mathbf{M}\omega_i^2\right)\phi_i = \mathbf{0}\,\forall i \quad \Leftrightarrow \quad \left(\mathbf{K} - \mathbf{M}\Omega^2\right)\mathbf{\Phi} = \mathbf{0}$$

Undamped Free Vibrations V

In practice (numerical approach)

Eigen vectors and eigen valeus are computed with the `eig` function:

$$[\Phi; \Omega^2] = \mathrm{eig}(\mathbf{K}, \mathbf{M})$$

Normalization of eigen vectors

Option 1: Normalization to a unit maximum absolute value, i.e. such that

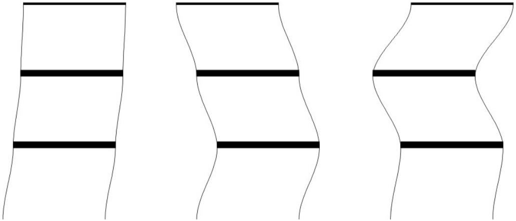
$$\max_j |\Phi_{ji}| = 1$$

Option 2: Normalization with respect to the mass, i.e. such that

$$\phi_i^T \mathbf{M} \phi_i = 1$$

NB: $\Phi_{ji} \equiv$ mode i at dof j

Undamped Free Vibrations VI



Mode 1 - f=1.66Hz

Mode 2 - f=4.53Hz

Mode 3 - f=6.19Hz

$$\Omega^2 = \begin{bmatrix} 108.5 & & \\ & 809.5 & \\ & & 1510.6 \end{bmatrix} \rightarrow \omega = \begin{Bmatrix} 10.4 \\ 28.5 \\ 38.9 \end{Bmatrix} \text{rad/s} \rightarrow f = \begin{Bmatrix} 1.66 \\ 4.53 \\ 6.19 \end{Bmatrix} \text{Hz}$$

$$\Phi = \begin{bmatrix} 0.5 & 1 & 0.5 \\ 0.866 & 0 & -0.866 \\ 1 & -1 & 1 \end{bmatrix}$$

Undamped Free Vibrations VII

Properties of eigen modes

Eigen modes are orthogonal through the mass and stiffness matrices

Demo: Let's consider two modes i and j with different frequencies ($\omega_i \neq \omega_j$)

$$\begin{aligned} \phi_i^T \mathbf{K} \phi_j &= \omega_j^2 \phi_i^T \mathbf{M} \phi_j \\ \phi_j^T \mathbf{K} \phi_i &= \omega_i^2 \phi_j^T \mathbf{M} \phi_i \end{aligned} \qquad (47)$$

$$\rightarrow \phi_i^T \mathbf{K} \phi_j - \phi_j^T \mathbf{K} \phi_i = \omega_j^2 \phi_i^T \mathbf{M} \phi_j - \omega_i^2 \phi_j^T \mathbf{M} \phi_i$$

Because $\phi_i^T \mathbf{K} \phi_j = \phi_j^T \mathbf{K} \phi_i$ and $\phi_i^T \mathbf{M} \phi_j = \phi_j^T \mathbf{M} \phi_i$,

$$0 = \left(\omega_j^2 - \omega_i^2\right) \phi_j^T \mathbf{M} \phi_i$$

so that $\phi_j^T \mathbf{M} \phi_i = \phi_i^T \mathbf{M} \phi_j = 0$. Backsubstituting into (47), $\phi_j^T \mathbf{K} \phi_i = \phi_i^T \mathbf{K} \phi_j = 0$.

(nb: one can demonstrate the same relations for $\omega_i = \omega_j$, but $\phi_i \neq \phi_j$)

Undamped Free Vibrations VIII

Modal Properties of the Structure

Important Property:

$$\mathbf{M}^* = \boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} \text{ and } \mathbf{K}^* = \boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} \text{ are diagonal matrices}$$

- Generalized Mass Matrix

$$\mathbf{M}^* = \boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi}$$

- Generalized Stiffness Matrix

$$\mathbf{K}^* = \boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi}$$

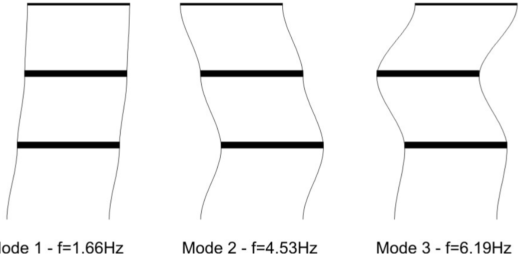
- Generalized Damping Matrix

$$\mathbf{C}^* = \boldsymbol{\Phi}^T \mathbf{C} \boldsymbol{\Phi}$$

→ represent the properties of the structure in the basis of normal modes

Undamped Free Vibrations IX

Example



$$\mathbf{M}^* = \begin{bmatrix} 4500 & & \\ & 4500 & \\ & & 4500 \end{bmatrix} (\text{kg}); \quad \mathbf{K}^* = \begin{bmatrix} 0.488 & & \\ & 3.64 & \\ & & 6.80 \end{bmatrix} .10^6 (\text{N/m})$$

Approximate Estimation of Natural Frequencies I

Rayleigh Quotient

The Rayleigh Quotient of a vector $\mathbf{y}$ is defined as

$$R(\mathbf{y}) = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

1. The Rayleigh quotient of a mode shape is equal to the squared natural frequency
$$R(\phi_i) = \frac{\phi_i^T \mathbf{K} \phi_i}{\phi_i^T \mathbf{M} \phi_i} = \omega_i^2$$
2. If $\mathbf{y}$ is a perturbation of ϕ_i of order ε, then $R(\mathbf{y})$ is an estimation of $R(\phi_i) = \omega_i^2$ with an error of order ε^2. In particular, if $\mathbf{y}$ is a perturbation of ϕ_1 of order ε, $R(\mathbf{y})$ gives an estimation *by excess* of ω_1^2.

In practice, the Rayleigh quotient is used to give an estiamation of natural frequencies.

Approximate Estimation of Natural Frequencies II

Rayleigh Quotient - Example

Let

$$\mathbf{y} = \langle 1/3 \quad 2/3 \quad 1 \rangle^T.$$

It is straightforward that

$$R(\mathbf{y}) = 127.8\,(rad/s)^2$$

The "corresponding" natural frequency is thus

$$f_{1,estimated} = \frac{1}{2\pi}\sqrt{R(\mathbf{y})} = 1.80 \text{ Hz}$$

(to be compared to $f_1 = 1.66$ Hz, ...
for a somewhat different mode shape $\phi_1 = \langle 0.5 \quad 0.866 \quad 1 \rangle^T$).

Iterative Approach

The fundamental frequency and the corresponding mode shape satisfy

$$\frac{1}{\omega_1^2}\mathbf{K}\,\phi_1 = \mathbf{M}\,\phi_1. \tag{48}$$

Let us assume that ϕ_1 is normalized to a maximum unit value.
It turns out that $\frac{1}{\omega_1^2}$ appears as the maximum static deflection under the "load" $\mathbf{M}\,\phi_1$...
...and ϕ_1 represents this static shape

Instead of formally satisfying (48), we can compute approached values $\hat{\omega}_1$ and $\hat{\phi}_1$ by considering

$$\frac{1}{\hat{\omega}_1^2}\mathbf{K}\,\hat{\phi}_1 = \mathbf{M}\,\overline{\phi_1} \tag{49}$$

where $\overline{\phi_1}$ is an approximation of the mode shape.

Approximate Estimation of Natural Frequencies IV

→ A **good estimation** of the natural frequency and a **better estimation** of an approximate mode shape are obtained as a result of a simple static computation

Approximate Estimation of Natural Frequencies V

Example

Simply supported beam (length L, bending stiffness EI, lineic mass μ)

The exact fundamental frequency is given by

$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\mu}} = \frac{9.87}{L^2}\sqrt{\frac{EI}{\mu}}$$

Let us assume $\overline{\phi_1} = 1$ (really tough, but at least normalized to 1) $\rightarrow \frac{1}{\hat{\omega}_1^2}\hat{\phi}_1$ represents the static deflection (i.e. deformed shape) under this uniformly distributed mass. The mid-span deflection is

$$\frac{1}{\hat{\omega}_1^2} = \frac{5\mu L^4}{384 EI} \quad \rightarrow \quad \hat{\omega}_1 = \frac{8.76}{L^2}\sqrt{\frac{EI}{\mu}}$$

and the shape is

$$\hat{\phi}_1 = \frac{16}{5}\left(\frac{x^4}{L^4} - 2\frac{x^3}{L^3} + \frac{x}{L}\right)$$

nb: $EI\frac{d^4 v}{dx^4} = \mu \quad \rightarrow \quad v = \frac{5\mu L^4}{384 EI}\frac{16}{5}\left(\frac{x^4}{L^4} - 2\frac{x^3}{L^3} + \frac{x}{L}\right)$

Approximate Estimation of Natural Frequencies VI

Instead of choosing $\overline{\phi_1} = 1$, we have a better estimation now

$$\overline{\phi_1} = \frac{16}{5}\left(\frac{x^4}{L^4} - 2\frac{x^3}{L^3} + \frac{x}{L}\right)$$

which produces

$$\frac{1}{\hat{\omega}_1^2} = \frac{277\mu L^4}{26880 EI} \quad \rightarrow \quad \hat{\omega}_1 = \frac{9.85}{L^2}\sqrt{\frac{EI}{\mu}}$$

$$\hat{\phi}_1 = \frac{256}{1385}\left(17\frac{x}{L} - 28\frac{x^3}{L^3} + 14\frac{x^5}{L^5} - 4\frac{x^7}{L^7} + \frac{x^8}{L^8}\right)$$

In practice, no iteration is performed and ...
the natural frequency corresponding to a mode shape is obtained by loading the structure with its own mass in the direction corresponding to the sign of the expected mode shape. If the maximum displacement is denoted by δ, the circular natural frequency is estimated by $\omega_1 = 1/\sqrt{\delta}$.

In short: it is always possible to determine (at least an approximation of) natural frequencies

Modal Synthesis I

The N natural modes ϕ_i form a basis of linearly independent vectors. Any vector $\mathbf{y}$ is expressed uniquely as a combination of these N independent vectors

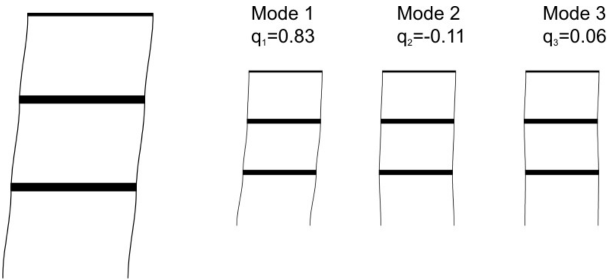
$$\mathbf{y} = \sum_{i=1}^{N} q_i \phi_i = \mathbf{\Phi}\,\mathbf{q}$$

where $\mathbf{\Phi}$ is the $N \times N$ matrix gathering the eigen vectors (modes). So,

$$\mathbf{q} = \mathbf{\Phi}^{-1}\mathbf{y}$$

Example

Modal Synthesis II



$$\begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} = 0.83 \cdot \begin{pmatrix} 0.5 \\ 0.866 \\ 1 \end{pmatrix} - 0.11 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 0.06 \cdot \begin{pmatrix} 0.5 \\ -0.866 \\ 1 \end{pmatrix}$$

Modal Synthesis III

Decomposition of the double pendulum

Modal Synthesis IV

Idea: solve the equation of motion with this decomposition. For each time step t:

$$\mathbf{x}(t) = \sum_{i=1}^{N} q_i(t)\,\phi_i = \mathbf{\Phi}\,\mathbf{q}(t)$$

Instead of solving

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t),$$

we solve

$$\mathbf{M\Phi}\ddot{\mathbf{q}}(t) + \mathbf{C\Phi}\dot{\mathbf{q}}(t) + \mathbf{K\Phi}\mathbf{q}(t) = \mathbf{p}(t) \tag{50}$$

$\rightarrow$ change of variable (unknown)

Modal Synthesis V

Idea: use advantage of $\mathbf{M}^* = \Phi^T \mathbf{M} \Phi$ and $\mathbf{K}^* = \Phi^T \mathbf{K} \Phi$ (diagonal matrices), and multiply (50) by Φ^T:

$$\Phi^T \mathbf{M} \Phi \ddot{\mathbf{q}}(t) + \Phi^T \mathbf{C} \Phi \dot{\mathbf{q}}(t) + \Phi^T \mathbf{K} \Phi \mathbf{q}(t) = \Phi^T \mathbf{p}(t)$$

or

$$\mathbf{M}^* \ddot{\mathbf{q}}(t) + \mathbf{C}^* \dot{\mathbf{q}}(t) + \mathbf{K}^* \mathbf{q}(t) = \mathbf{p}^*(t)$$

where $\mathbf{p}^*(t) = \Phi^T \mathbf{p}(t)$ represent *generalized forces.*

→ the equations of motion are projected in the modal basis

Modal Synthesis VI

Few information about structural damping: give $\mathbf{C}^*$ a simple form $\rightarrow$ diagonal!
The system

$$\mathbf{M}^*\ddot{\mathbf{q}}(t)+\mathbf{C}^*\dot{\mathbf{q}}(t)+\mathbf{K}^*\mathbf{q}(t)=\mathbf{p}^*(t)$$

is thus a concatenation of independent equations (uncoupled system).
A generic equation reads

$$M^*_{i,i}\ddot{q}_i(t)+C^*_{i,i}\dot{q}_i(t)+K^*_{i,i}q_i(t)=p^*_i(t)$$

or

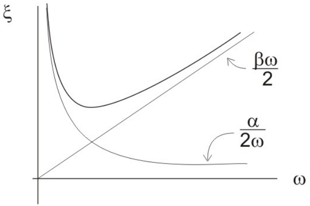
$$\ddot{q}_i(t)+2\omega_i\xi_i\dot{q}_i(t)+\omega_i^2 q_i(t)=\frac{p^*_i(t)}{M^*_{i,i}}$$

Because there is few information about damping, it is usually simply characterized by ξ_i $\rightarrow$ method well adapted to the modal basis approach.

Proportional Damping I

Another way to obtain a diagonal modal damping matrix is to assume

$$\mathbf{C}^* = \alpha \mathbf{M}^* + \beta \mathbf{K}^* \quad \text{i.e.} \quad \mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}$$



This choice has an influence on the damping coefficient in each mode:

$$\xi_i = \frac{C^*_{i,i}}{2M^*_{i,i,}\omega_i} = \frac{\alpha M^*_{i,i} + \beta K^*_{i,i}}{2M^*_{i,i,}\omega_i} = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

Rayleigh damping

Proportional Damping II

In practice α and β are determined in such a way to fix the damping ratio to a desired value for 2 modes

$$\begin{cases} \xi_i &= \dfrac{\alpha}{2\omega_i} + \dfrac{\beta\omega_i}{2} \\ \xi_j &= \dfrac{\alpha}{2\omega_j} + \dfrac{\beta\omega_j}{2} \end{cases} \Rightarrow \begin{cases} \alpha &= \dfrac{2\omega_i\omega_j}{\omega_i^2 - \omega_j^2}\left(\omega_i\xi_j - \omega_j\xi_i\right) \\ \beta &= \dfrac{2\omega_i\omega_j}{\omega_i^2 - \omega_j^2}\left(\dfrac{\xi_i}{\omega_j} - \dfrac{\xi_j}{\omega_i}\right) \end{cases}$$

(there's no way to impose a specific value for the other modes; one can just compute them...)
This technique is used in a nodal analysis where an expression of $\mathbf{C}$ is necessary.

If damping cannot be considered as diagonal,

- solve the coupled system (even in a modal basis thus)
- neglect off-diagonal elements

Modal Basis Analysis I

One has to solve

$$\ddot{q}_i(t) + 2\omega_i \xi_i \dot{q}_i(t) + \omega_i^2 q_i(t) = \frac{p_i^*(t)}{M_{i,i}^*} \quad \text{pour } i = 1, \ldots N$$

→ use the methods developed for the S-DOF system

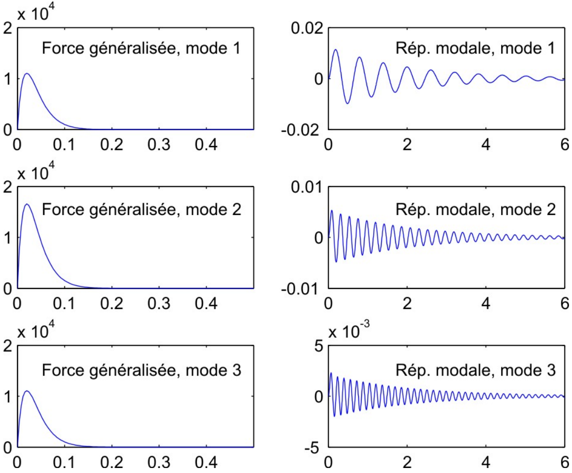
- time domain, Duhamel's convolution integral
- frequency domain (multiplication by the FRF function)
- step-by-step method (time domain)

Repeat the computation for each mode, to determine $q_i(t)$, $i = 1, \cdots, N$

Combine modal responses:

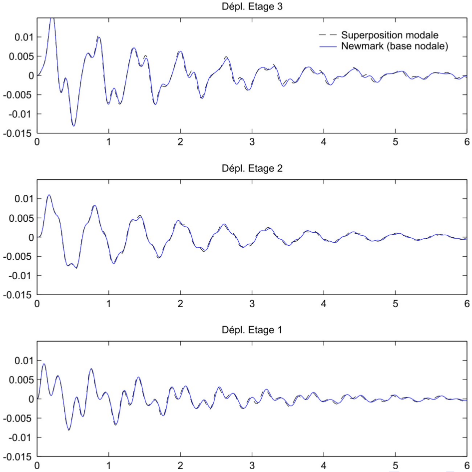
$$\mathbf{x}(t) = \sum_{i=1}^{N} q_i(t)\phi_i = \mathbf{\Phi q}(t)$$

Modal Basis Analysis II



Modal Basis Analysis III

Example - comparison with the solution obtained in the nodal basis



Modal Basis Analysis IV

Example - Duhamel's convolution integral

$$q(t) = \frac{1}{m\omega_d} \int_0^t p(\tau)\, e^{-\xi\omega_1(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

Idea ? Limit the computation to the first M modes (neglect higher modes for which $\omega_d \gg$)

$$\mathbf{x}(t) = \sum_{i=1}^{N} q_i(t)\,\phi_i \simeq \sum_{i=1}^{M} q_i(t)\,\phi_i = \Phi\,\mathbf{q}(t)$$

where Φ represents now an $N \times M$ *rectangular* matrix and $\mathbf{q}$ is the vector gathering the M modal coordinates.

In practice, on can keep only a list of M modes (not necessarily the first M ones).

Modal Basis Analysis V

How to select the relevant modes for a given dynamic analysis ?

Context: a load with a frequency content in a limited band $[0, \Omega_{soll,\max}]$
Frequency Domain Analysis:

$$\mathscr{Q}(\Omega) = \mathscr{H}(\Omega)\,\mathscr{P}^*(\Omega)$$

Neglect modes for which $\mathscr{H}(\Omega)\,\mathscr{P}^*(\Omega)$ is small:

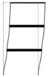
- either $\mathscr{H}(\Omega)$ is small in $[0, \Omega_{soll,\max}]$ (argument 1: $K^*_{i,i}$ is usually a decreasing series // argument 2: quasi-static response if $f_{nat} \gtrsim 5 f_{soll,\max}$;
- either $\mathscr{P}^*(\Omega)$ is small for the considered loading (Example 1: simply supported beam with symmetrical loading // Example 2: the frame)

Other possibility:

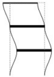
- $\mathscr{Q}(\Omega)$ is not negligible, but the considered mode has few influence on the quantity that is observed

Modal Basis Analysis VI

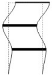
Mode 1 - f=1.66Hz



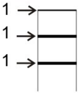
Mode 2 - f=4.53Hz



Mode 3 - f=6.19Hz

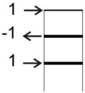


Exemple 1



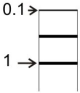
$$P^*=\begin{pmatrix} 2.37 \\ 0 \\ 0.63 \end{pmatrix}$$

Exemple 2



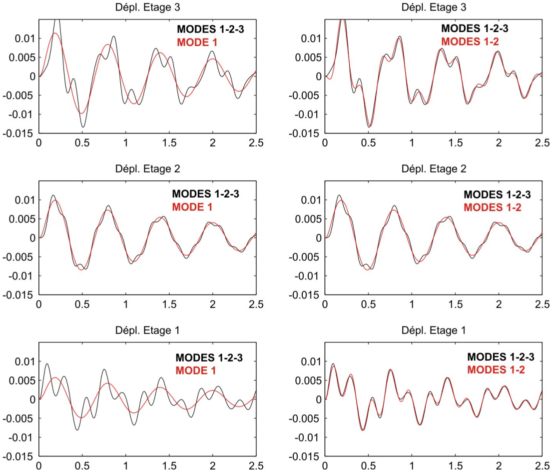
$$P^*=\begin{pmatrix} 0.6 \\ 0 \\ 2.37 \end{pmatrix}$$

Exemple 3



$$P^*=\begin{pmatrix} 0.6 \\ 0.9 \\ 0.6 \end{pmatrix}$$

Modal Basis Analysis VII



In very short...

Dynamic analysis in the modal basis is interesting because

- it decreases the number of unknowns ($M \ll N$ in practice)
- it allows a decoupling of modal equations

The most popular analysis method is a step-by-step analysis in the modal basis.

Section V: Continuous Structures

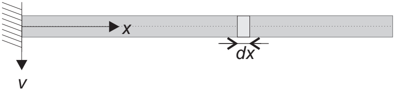
Learning Outcomes:

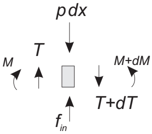
- rapid dynamic analysis of simple structures
- estimation of natural frequencies of continuous systems
- mathematical treatment of PDE with eigen functions

Transverse Vibrations I

Continuous Version

Example: Transverse vibrations of a Bernoulli beam





$$\mu \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v(x,t)}{\partial x^2} \right) = p(x,t)$$

with μ the lineic mass (mass per unit length) and EI the bending stiffness.

$$T = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI \frac{\partial^2 v}{\partial x^2} \right) \qquad \text{and} \qquad f_i = \mu\, dx \frac{\partial^2 v}{\partial t^2}$$

[Beam equation, with inertial forces]

Transverse Vibrations II

How to get rid of the PDE ?

Lumped modeling → 1-DOF

1. assume the response takes place in a given shape $v(x,t) = \phi(x)\, q(t)$ ($\phi(x)$ satisfies boundary conditions)
2. project the response in the assumed shape

$$m^\star \ddot{q}(t) + k^\star q(t)\, dx = \int_\Omega \phi(x)\, p(x,t)\, dx$$

with $m^\star = \int_\Omega \mu \phi^2(x)\, dx$ and $k^\star = \int_\Omega \phi(x) \frac{d^2}{dx^2}\left(EI \frac{d^2 v(x,t)}{dx^2}\right) dx$.

[nb: Another way to cope with the difficulty of the PDE is to recourse to eigen functions - See section related to vibrations of continuous systems]

[nb: Another alternative is the finite element method (transformation of an ODE into a set of algebraic equations - See Discrete Version]

Normal modes of vibration I

Aim: to write the governing equation in a modal basis

→ Procedure is similar to what is done for discrete MDOF (modal analysis)

We shall look for a particular solution of the equation of motion (without loading) under the separation of variable format

$$v(x,t) = \phi(x)\, q(t)\,.$$

The introduction of this particular solution format into the equation of motion yields

$$\phi(x)\,\ddot{q}(t) + \frac{EI}{\mu}\phi''''(x)\, q(t) = 0,$$

or, equivalently,

$$-\frac{\ddot{q}(t)}{q(t)} = \frac{EI}{\mu}\frac{\phi''''(x)}{\phi(x)}$$

Normal modes of vibration II

As both sides of this equation are functions of different variables, the only way to satisfy this equation is to have both sides of the equation equal to the same (positive) constant

$$-\frac{\ddot{q}(t)}{q(t)} = \frac{EI}{\mu}\frac{\phi''''(x)}{\phi(x)} = \omega^2$$

Actually we will show that ω can't take any value, but well a set of well-determined values.
[nb: upper dot stands for time derivative, whereas prime symbol stands for space gradient.]

Mode shapes are obtained from

$$\phi''''(x) - \frac{\mu\omega^2}{EI}\phi(x) = 0 \qquad (51)$$

and time evolution of modal amplitudes are obtained from

$$\ddot{q}(t) + \omega^2 q(t) = 0$$

nb: this latter equation gives a meaning to ω

Normal modes of vibration III

Mode shapes are determined from (51), and require knowledge of boundary conditions. The general solution of (51) is

$$\phi(x) = A \sin kx + B \cos kx + C \sinh kx + D \cosh kx$$

where $k^4 = \frac{\mu \omega^2}{EI}$.

Normal modes of vibration IV

Example: simply supported beam

In this example, the end conditions are

$$\begin{aligned} \phi(0) &= \phi(L) = 0 \\ \phi''(0) &= \phi''(L) = 0 \end{aligned}$$

Satisfaction of these conditions yields

$$B = 0 \quad ; \quad D = 0 \quad ; \quad A \sin kL = 0 \quad ; \quad C \sinh kL = 0$$

The only non-trivial solution is $A \sin kL = 0$, which results in $kL = i\pi$, i.e.

$$\omega_i = \left(\frac{i\pi}{L}\right)^2 \sqrt{\frac{EI}{\mu}}$$

$$\phi_i(x) = A \sin \frac{i\pi x}{L} \tag{52}$$

nb: the mode shape is normalized by setting a unit maximum value, i.e. $A = 1$ (in each mode).

Normal modes of vibration V

Orthogonality of Normal Modes

Orthogonality is not demonstrated - as such - (see discrete form of equation of motion).

It is possible to show that

$$\int_0^L \mu \phi_k(x) \phi_i(x) \, dx = M_i^* \delta_{ki}$$

$$\int_0^L EI \phi_k(x) \phi_i''''(x) \, dx = K_i^* \delta_{ki}$$

(no matter the boundary conditions)

Normal modes of vibration VI

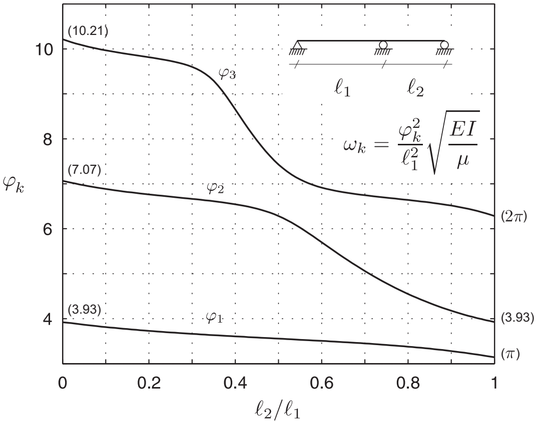
The first few mode shapes and natural frequencies may be computed with this technique. Tables provide estimations of the natural frequencies with simple (exact) formula.

Example:

Normal modes of vibration VII

Transverse vibrations		Longitudinal vibrations
$$\omega_k = \frac{\varphi_k^2}{\ell^2}\sqrt{\frac{EI}{\mu}}$$		$$\omega_k = \frac{\varphi_k}{\ell}\sqrt{\frac{E}{\rho}}$$
		
$\varphi_1 = \pi$ $\varphi_2 = 2\pi$ $\varphi_3 = 3\pi$ $\varphi_4 = 4\pi$	$\varphi_1 = 4.73$ $\varphi_2 = 7.85$ $\varphi_3 = 11.0$ $\varphi_4 = 14.1$	$\varphi_k = k\,\pi$
		
$\varphi_1 = 1.88$ $\varphi_2 = 4.69$ $\varphi_3 = 7.85$ $\varphi_4 = 11.0$	$\varphi_1 = 3.93$ $\varphi_2 = 7.07$ $\varphi_3 = 10.2$ $\varphi_4 = 13.3$	$\varphi_k = \frac{\pi}{2} + k\,\pi$

NORMAL MODES OF VIBRATION VIII



Modal Basis Analysis I

The solution of the equation of motion is obtained with this change of variable

$$v(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\, q_i(t)$$

where q_i are the modal amplitudes (new unknowns of the problem). The equation of motion in the modal space is obtained by "left-multiplication" by the mode shape

$$\int_0^L \phi_k(x) \left[\mu \frac{\partial^2 v(x,t)}{\partial t^2} + EI \frac{\partial^4 v(x,t)}{\partial x^4} = p(x,t) \right] dx$$

and thus reads

$$M_k^* \ddot{q}_k(t) + K_k^* q_k(t) = \int_0^L \phi_k(x)\, p(x,t)\, dx = p_k^*(t) \quad \text{for } k = 1, \ldots \infty$$

with [definition of generalized mass and stiffness]

$$\int_0^L \mu \phi_k(x)\, \phi_i(x)\, dx = M_i^* \delta_{ki}$$

$$\int_0^L EI \phi_k(x)\, \phi_i''''(x)\, dx = K_i^* \delta_{ki}$$

Modal Basis Analysis II

The modal equations are uncoupled

$$M_k^* \ddot{q}_k(t) + K_k^* q_k(t) = \int_0^L \phi_k(x)\, p(x,t)\, dx = p_k^*(t) \quad \text{for } k = 1, \ldots \infty$$

→ transformation to a set of ODE (instead of a PDE). Possible to digitalize the solution (provided a truncation on k is performed).

nb: we have introduced p_k^*, the generalized force in mode k.

Modal Basis Analysis III

Example: simply supported beam

Generalized masses and stiffnesses are readily obtained as (*)

$$\begin{aligned} M_i^* &= \int_0^L \mu \sin^2 \frac{i\pi x}{L} dx = \frac{\mu L}{2} \\ K_i^* &= \int_0^L EI \left(\frac{i\pi}{L}\right)^4 \sin^2 \frac{i\pi x}{L} dx = EI \left(\frac{i\pi}{L}\right)^4 \frac{L}{2} \end{aligned} \tag{53}$$

We can also check that

$$\frac{K_i^*}{M_i^*} = \frac{EI\left(\frac{i\pi}{L}\right)^4 \frac{L}{2}}{\frac{\mu L}{2}} = \omega_i^2$$

which is comfortable to justify the physical meaning we gave to $\boldsymbol{\omega}$.

(*) because $\int_0^L \sin \frac{i\pi x}{L} \sin \frac{j\pi x}{L} dx = \frac{L}{2} \delta_{ij}$

Modal Basis Analysis IV

Refinement with damping

In this solution procedure, damping is added *a posteriori*, by means of damping coefficients

$$M_k^* \ddot{q}_k(t) + 2\xi_k \sqrt{K_k^* M_k^*}\, \dot{q}_k(t) + K_k^* q_k(t) = p_k^*(t)$$

where damping coefficients ξ_k are selected according to the constitutive material of the structure.

Summary: *typical analysis outline*

1. Compute mode shapes and natural frequencies
2. Compute generalized masses ans stiffnesses, as well as generalized forces
3. Solve (independently!) for the modal amplitude in each mode (e.g. simple 1-DDL Newmark approach)
4. Return to nodal displacements by means of the initial change of variables $v(x,t) = \sum_{i=1}^{M} \phi_i(x)\, q_i(t)$ (where M is the number of modes considered in the analysis)

Application to bridge crossing I

Méca II, 2017-2018

V. Denoël

Problem: A force with constant amplitude F crosses a simply supported beam (given EI, μ and L) at a given velocity v.

---

1. Modes shapes and natural frequencies are given by (52)

2. Generalized masses and stiffnesses are given by (53)
The force is located at $x = 0$ at $t = 0$ and therefore at $x = v\,t$ at time t. The distributed load applied on the beam is thus given as

$$p(x,t) = F\,\delta(x - v\,t) \tag{54}$$

The generalized forces are obtained as

$$p_k^*(t) = \int_0^L \phi_k(x)\,p(x,t)\,dx = F\,\phi_k(v\,t) = F\sin\frac{k\pi v\,t}{L} \tag{55}$$

3. The response in each mode is obtained by solving (undamped structure)

$$M_k^*\ddot{q}_k(t) + K_k^* q_k(t) = F\sin\frac{k\pi v\,t}{L}$$

After a bit of algebra...

Application to bridge crossing II

$$q_k(t) = \frac{F}{K_k^*} \frac{\sin \frac{k\pi t v}{L} - \frac{k\pi v}{\omega_k L} \sin \omega_k t}{1 - \left(\frac{k\pi v}{\omega_k L}\right)^2}$$

or, noticing that $\omega_k = k^2 \omega_1$ and after introduction of $\alpha = \frac{\pi v}{\omega_1 L}$ (dimensionless velocity of the vehicle)

$$q_k(t) = \frac{F}{K_k^*} \frac{\sin\left(\frac{\alpha}{k}\omega_k t\right) - \frac{\alpha}{k}\sin(\omega_k t)}{1 - \left(\frac{\alpha}{k}\right)^2} \quad (56)$$

→ danger in mode k if $\alpha \simeq k$ (nb: in practical civil engineering applications $\alpha \ll 1$).

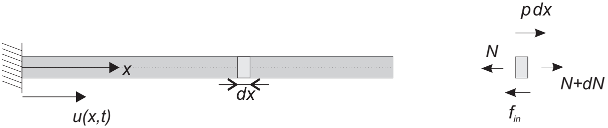
4. Return to nodal displacements. For instance, at mid-span

$$v\left(\frac{L}{2}, t\right) = \sum_{k=1;k \text{ odd}}^{M} (-1)^k \frac{F}{K_k^*} \frac{\sin\left(\frac{\alpha}{k}\omega_k t\right) - \frac{\alpha}{k}\sin(\omega_k t)}{1 - \left(\frac{\alpha}{k}\right)^2}$$

Longitudinal Vibrations I

Another example of Continuous Model

Longitudinal vibrations of a (truss) bar



$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = E \frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t)$$

with ρ the material density (mass per unit volume) and E the Young modulus.

Mode shapes ?

$$u(x,t) = \phi(x)\, q(t) \qquad \rightarrow \qquad \phi(x)\, \ddot{q}(t) = \frac{E}{\rho} \phi''(x)\, q(t),$$

Longitudinal Vibrations II

from which

$$-\frac{\ddot{q}(t)}{q(t)} = -\frac{E}{\rho}\frac{\phi''(x)}{\phi(x)} = \omega^2$$

- mode shapes are obtained from

$$\phi''(x) + \frac{\rho\omega^2}{E}\phi(x) = 0 \qquad \rightarrow \qquad \phi(x) = A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}$$

with $c = \sqrt{E/\rho}$.

- time evolution of modal amplitudes are obtained from

$$\ddot{q}(t) + \omega^2 q(t) = 0 \qquad \rightarrow \qquad q(t) = C\sin\omega t + D\cos\omega t$$

Example

Fixed ends: $u(0) = u(L) = 0$.
In this case,

$$\sin\frac{\omega x}{c} = 0 \qquad \rightarrow \qquad \omega_i = i\pi\frac{c}{L} = \frac{i\pi}{L}\sqrt{\frac{E}{\rho}}$$

Section VI: Seismic Analysis

Learning Outcomes:

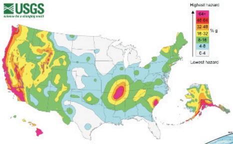
- origin of earthquakes and seismic risk
- to have the required analysis tools for seismic engineering

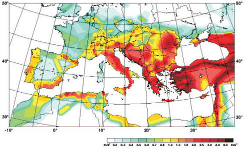
Earthquakes I

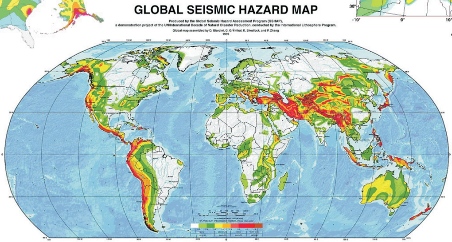
Why seismic engineering ?

Earthquakes II

Earthquakes III





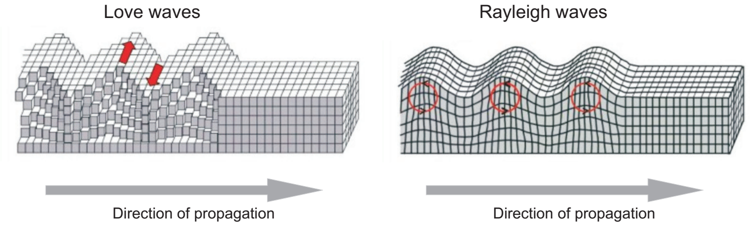


The seismic hazard maps provide earthquake properties, such as peak ground acceleration (PGA), associated with a given return period, usually 475 years (probability of exceedance of 10% over 50 years) and 95 years (probability of exceedance of 10% over 10 years).

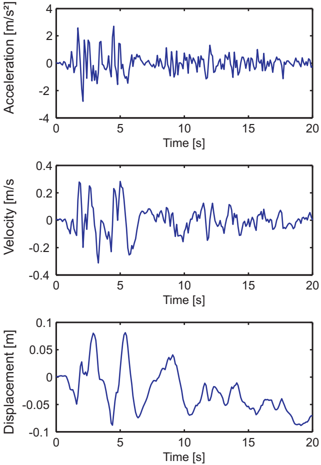
Earthquakes IV

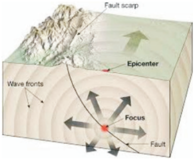
Forces induced by impulsive loading in a semi-infinite half-space are composed of:

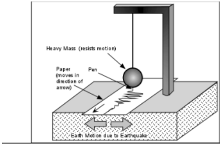
- body waves: P-wave and S-waves
- surface waves: Rayleigh and Love waves



Earthquakes V







Earthquakes VI

The most important information for a structural and civil engineer is the **accelerogram** (or **seismogram**).

Earthquakes VII

Characterization of seismic activity

Obsolete / Useless charaterizations:

- **Duration** T [s], usually $\in [10; 60]$ seconds - {the longer the worse}
- **Intensity** I - Mercalli Scale - {is that well interesting ?}
- **Magnitude** M - Richter Scale - ($\log E = 11.8 + 1.5M$), with E the energy at the **focus** - {few interest because of that}
- **Maximum displacement** [m], usually $\in [0.01; 1]$ m - {is that well interesting ?}
- **Maximum acceleration** [m/s^2], usually $\in [0; 1]$ m/s^2 - {better: related to effective loading}

Appropriate charaterizations:

- **Accelerogram** [m/s^2] - {much better: solution of equation of motion}
- **Response Spectra** $S(T_1, \xi_1)$ - even better: for lazy engineers

Earthquakes VIII

The Mercalli Scale

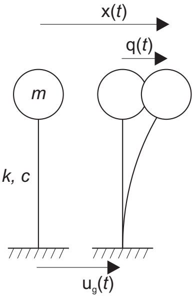
I. Instrumental Generally not felt by people unless in favorable conditions.
II. Weak Felt only by a couple people that are sensitive, especially on the upper floors of buildings. Delicately suspended objects (including chandeliers) may swing slightly.
III. Slight Felt quite noticeably by people indoors, especially on the upper floors of buildings. Many do not recognize it as an earthquake. Standing automobiles may rock slightly. Vibration similar to the passing of a truck. Duration can be estimated. Indoor objects (including chandeliers) may shake.
IV. Moderate Felt indoors by many to all people, and outdoors by few people. Some awakened. Dishes, windows, and doors disturbed, and walls make cracking sounds. Chandeliers and indoor objects shake noticeably. The sensation is more like a heavy truck striking building. Standing automobiles rock noticeably. Dishes and windows rattle alarmingly. Damage none.
V. Rather Strong Felt inside by most or all, and outside. Dishes and windows may break and bells will ring. Vibrations are more like a large train passing close to a house. Possible slight damage to buildings. Liquids may spill out of glasses or open containers. None to a few people are frightened and run outdoors.
VI. Strong Felt by everyone, outside or inside; many frightened and run outdoors, walk unsteadily. Windows, dishes, glassware broken; books fall off shelves; some heavy furniture moved or overturned; a few instances of fallen plaster. Damage slight to moderate to poorly designed buildings, all others receive none to slight damage.

Earthquakes IX

VII. Very Strong Difficult to stand. Furniture broken. Damage light in building of good design and construction; slight to moderate in ordinarily built structures; considerable damage in poorly built or badly designed structures; some chimneys broken or heavily damaged. Noticed by people driving automobiles.
VIII. Destructive Damage slight in structures of good design, considerable in normal buildings with a possible partial collapse. Damage great in poorly built structures. Brick buildings easily receive moderate to extremely heavy damage. Possible fall of chimneys, factory stacks, columns, monuments, walls, etc. Heavy furniture moved.
IX. Violent General panic. Damage slight to moderate (possibly heavy) in well-designed structures. Well-designed structures thrown out of plumb. Damage moderate to great in substantial buildings, with a possible partial collapse. Some buildings may be shifted off foundations. Walls can fall down or collapse.
X. Intense Many well-built structures destroyed, collapsed, or moderately to severely damaged. Most other structures destroyed, possibly shifted off foundation. Large landslides.
XI. Extreme Few, if any structures remain standing. Numerous landslides, cracks and deformation of the ground.
XII. Catastrophic Total destruction – everything is destroyed. Lines of sight and level distorted. Objects thrown into the air. The ground moves in waves or ripples. Large amounts of rock move position. Landscape altered, or leveled by several meters. Even the routes of rivers can be changed.

Response of a 1-DOF Oscillator I

Governing Equations



$$m\ddot{x}(t) + c\dot{q}(t) + kq(t) = 0 \quad \rightarrow \quad m\ddot{q}(t) + c\dot{q}(t) + kq(t) = -m\ddot{u}_g(t)$$

As if there was an effective loading $p_{eff} = -m\ddot{u}_g(t)$ in the *relative* reference frame.

Response of a 1-DOF Oscillator II

$$\ddot{q}(t) + 2\omega_1 \xi_1 \dot{q}(t) + \omega_1^2 q(t) = -\ddot{u}_g(t)$$

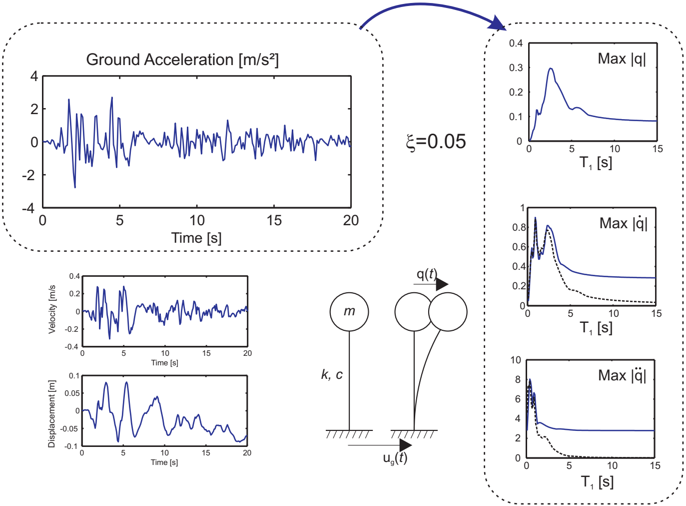
For a given accelerogram $\ddot{u}_g$, the response $q(t)$ depends on ω_1 and ξ_1 (the properties of the structure).

$$\ddot{q}(t) + \frac{4\pi}{T_1} \xi_1 \dot{q}(t) + \frac{4\pi^2}{T_1^2} q(t) = -\ddot{u}_g(t)$$

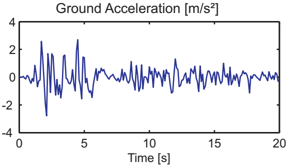
For a given accelerogram $\ddot{u}_g$, the response $q(t)$ depends on ω_1 and T_1 (the properties of the structure).

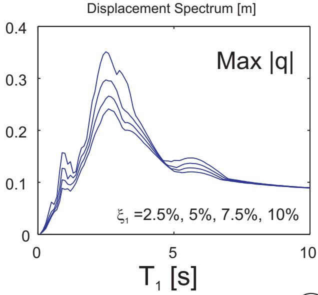
→ concept of **Response Spectrum**

Response of a 1-DOF Oscillator III



Response of a 1-DOF Oscillator IV



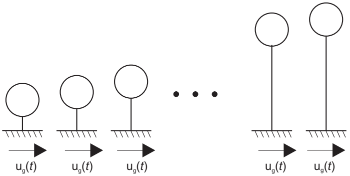


Inertial asymptote

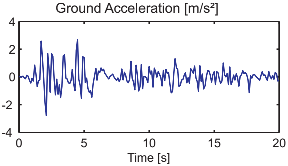
$$\lim_{T_1 \to +\infty} S_d(T_1) = u_{g,max}$$

Quasi-static asymptote

$$\lim_{T_1 \to 0} S_d(T_1) = \lim_{T_1 \to 0} \frac{-T_1^2}{4\pi^2} \ddot{u}_{g,max} = 0$$



Response of a 1-DOF Oscillator V



Inertial asymptote

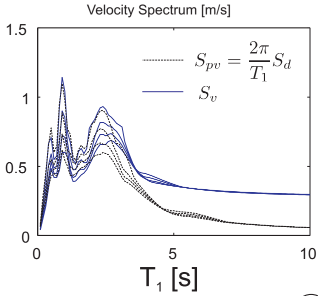
$$\lim_{T_1 \to +\infty} S_v(T_1) = \dot{u}_{g,max}$$

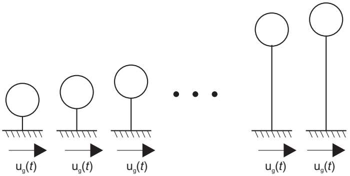
$$\lim_{T_1 \to +\infty} S_{pv}(T_1) = 0$$

Quasi-static asymptote

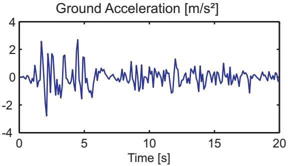
$$\lim_{T_1 \to 0} S_v(T_1) = \lim_{T_1 \to 0} \frac{-T_1^2}{4\pi^2} \ddot{u}_{g,max} = 0$$

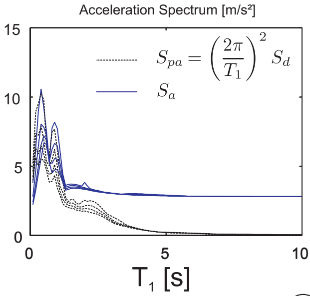
$$\lim_{T_1 \to 0} S_{pv}(T_1) = 0$$





Response of a 1-DOF Oscillator VI





Inertial asymptote

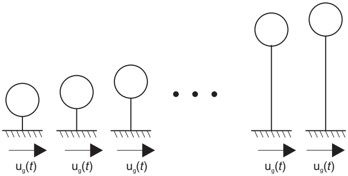
$$\lim_{T_1 \to +\infty} S_a(T_1) = \ddot{u}_{g,max}$$

$$\lim_{T_1 \to +\infty} S_{pa}(T_1) = 0$$

Quasi-static asymptote

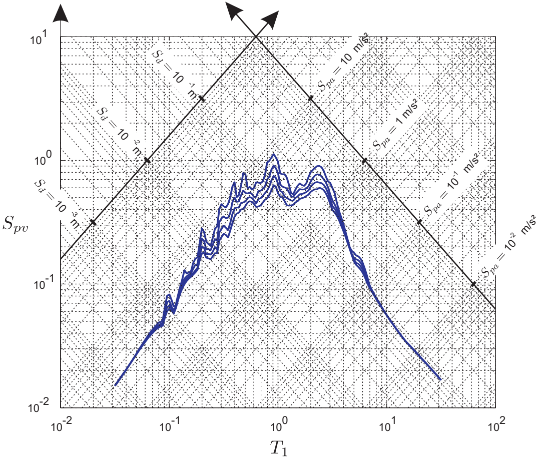
$$\lim_{T_1 \to 0} S_a(T_1) = \lim_{T_1 \to 0} \frac{-T_1^2}{4\pi^2} \ddddot{u}_{g,max} = 0$$

$$\lim_{T_1 \to 0} S_{pa}(T_1) = \ddot{u}_{g,max}$$

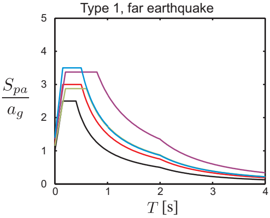


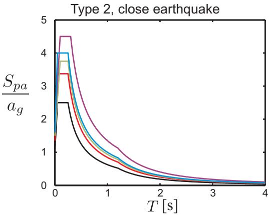
Response of a 1-DOF Oscillator VII

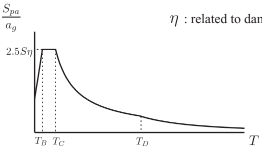
The Displacement, Pseudo-Velocity and Pseudo-Acceleration Response Spectra



Eurocode 8 - Spectra I







η : related to damping

$$
\begin{array}{ll}
a_g S \left(1 + \frac{T}{T_B}(2.5\eta - 1)\right) & \text{for } 0 \leq T \leq T_B \\
2.5 a_g S \eta & \text{for } T_B \leq T \leq T_C \\
2.5 a_g S \eta \frac{T_C}{T} & \text{for } T_C \leq T \leq T_D \\
2.5 a_g S \eta \frac{T_C T_D}{T^2} & \text{for } T_D \leq T
\end{array}
$$

Subsoil	S		T_B [s]		T_C [s]		T_D [s]	
	1	2	1	2	1	2	1	2
A	1.0	1.0	0.15	0.05	0.40	0.25	2.0	1.2
B	1.2	1.35	0.15	0.05	0.50	0.25	2.0	1.2
C	1.15	1.50	0.20	0.10	0.60	0.25	2.0	1.2
D	1.35	1.80	0.20	0.10	0.80	0.30	2.0	1.2
E	1.40	1.60	0.15	0.05	0.50	0.25	2.0	1.2

Parameters of the model

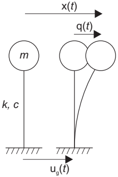
Eurocode 8 - Spectra II

Definition of subsoil classes according to EC 8 [1]

Subsoil	Description of stratigraphic profile	Parameters		
		$V_{s,30}$ [m/s]	N_{SPT} (blows/30cm)	c_u [kPa]
A	Rock or other rock-like geological formation, including at most 5 m of weaker material at the surface	> 800	-	-
B	Deposits of very dense sand, gravel, or very stiff clay, at least several tens of meters in thickness and characterized by a gradual increase of mechanical properties with depth	360 - 800	> 50	> 250
C	Deep deposits of dense or medium-dense sand, gravel, or stiff clay with thicknesses from several tens to many hundreds of meters	180 - 360	15 - 50	70 - 250
D	Deposits of loose-to-medium noncohesive soil (with or without some soft cohesive layers), or of predominantly soft-to-firm cohesive soil	< 180	< 15	< 70
E	Soil profile consisting of a surface alluvium layer with $V_{s,30}$ values of type C or D, and thicknesses varying between 5 m and 20 m, underlain by stiffer materials with $V_{s,30}$ > 800 m/s			
S_1	Deposits consisting or containing a layer at least 10 m thick of soft clays/silts with high plasticity index (PI > 40) and high water content	< 100	-	10 -20
S_2	Deposits of liquefiable soils, sensitive clays, or any other soil profile not included in types A-E or S_1			

Synthetic Accelerograms I

Analysis Options



$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = -m\ddot{u}_g(t)$$

- use effective loading $p_{eff} = -m\ddot{u}_g(t)$ in the *relative* reference frame, with
  - $\ddot{u}_g(t)$ from existing records at the same location (sometimes stretched/shrinked)
  - synthetic accelerograms corresponding to:
    - local wave propagation from bedrock to surface
    - standardized response spectrum
- use response spectrum → equivalent static loading $F_{equiv} = m \cdot S_{pa}(T)$

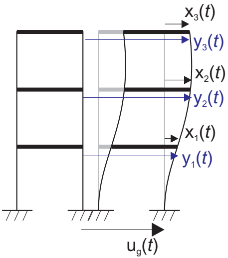
! limitations on the linearity of the structural behaviour !

Exercises

1. Compute the response spectra of the synthetic accelerogram you were given
2. Determine with which subsoil class and PGA this accelerogram could be associated

Response of an M-DOF Oscillator I

Governing Equations



$$\mathbf{M}\ddot{\mathbf{y}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=0 \quad \rightarrow \quad \mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=-\mathbf{M}\mathbf{r}\ddot{u}_g(t)$$

As if there was an effective loading $\mathbf{p}_{eff}=-\mathbf{M}\mathbf{r}\ddot{u}_g(t)$ in the *relative* reference frame.

Response of an M-DOF Oscillator II

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=-\mathbf{M}\mathbf{r}\ddot{u}_g(t) \tag{57}$$

- If accelerogram $\ddot{u}_g$ is available (from existing records at same location, from wave propagation analysis) → use effective loading $\mathbf{p}_{eff}=-\mathbf{M}\mathbf{r}\ddot{u}_g(t)$ and solve with time stepping
- If response spectrum is available
  - generate spectum compatible synthetic accelerograms (time stepping)
  - design with response spectrum (no solution of equation of motion is required)

< see MDOF systems & Newmark methods for time stepping >

Response of an M-DOF Oscillator III

Design of an MDOF structure with a response spectrum

Response spectrum → use modal basis

Let $\mathbf{x} = \mathbf{\Phi q}$, with the normal modes of vibration $\mathbf{\Phi}$,

$$\mathbf{M}^{\star}\ddot{\mathbf{q}}(t) + \mathbf{C}^{\star}\dot{\mathbf{q}}(t) + \mathbf{K}^{\star}\mathbf{q}(t) = -\mathbf{\Phi}^{T}\mathbf{Mr}\ddot{u}_g(t)$$

The modal response in mode i is given by (diagonal modal damping)

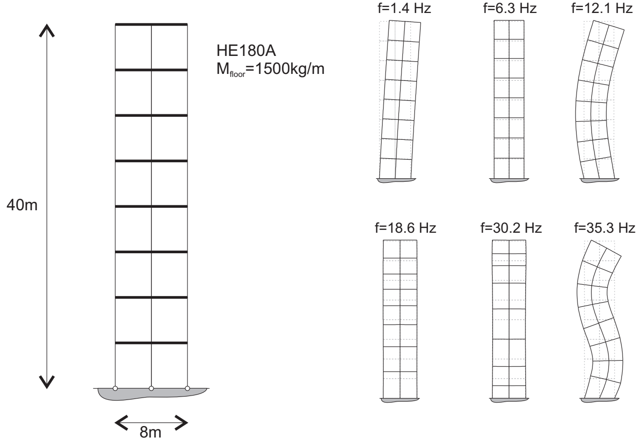
$$M_{ii}^{\star}\ddot{q}_i(t) + C_{ii}^{\star}\dot{q}_i(t) + K_{ii}^{\star}q_i(t) = \mathscr{L}_i\ddot{u}_g(t)$$

with $\mathscr{L} = -\mathbf{\Phi}^{T}\mathbf{Mr}$, the vector of ***modal participating masses***.
We also define the ***modal participating ratio***

$$\mu_i = \frac{1}{M_r}\frac{\mathscr{L}_i^2}{M_{ii}}$$

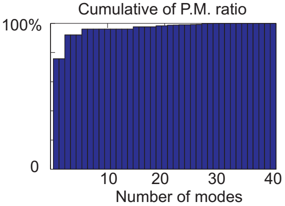
with $M_r = \mathbf{r}^{T}\mathbf{Mr}$, and the important property that $\sum_i \mu_i = 1$.

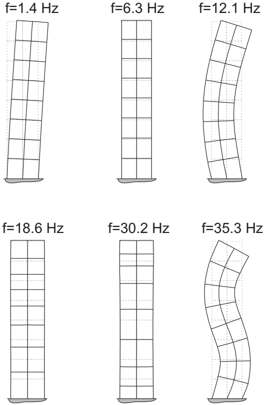
Response of an M-DOF Oscillator IV



Response of an M-DOF Oscillator V

Mode	$M_i^\star$[to]	$\mathcal{L}_i$[to]	μ_i[%]
1	37.5	-53.3	75.8%
2	53.6	0	0
3	49.5	28.4	16.3%
4	53.5	0	0
5	53.0	0	0
6	52.3	-14.5	4.03%





Response of an M-DOF Oscillator VI

We need to solve

$$M_{ii}^{\star}\ddot{q}_i(t) + C_{ii}^{\star}\dot{q}_i(t) + K_{ii}^{\star}q_i(t) = \mathcal{L}_i\ddot{u}_g(t)$$

The response spectrum $S_d(T_1)$ was defined as the maximum (abs.) of $q(t)$ from

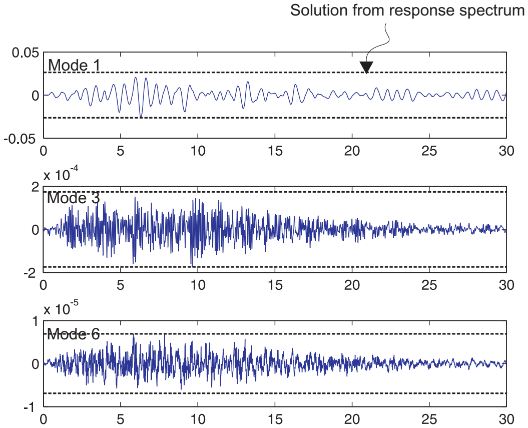
$$\ddot{q}(t) + 2\omega_1\xi_1\dot{q}(t) + \omega_1^2 q(t) = -\ddot{u}_g(t)$$

Comparison of these two thus shows that

$$q_{i,max} = \max_t |q_i(t)| = \frac{|\mathcal{L}_i|}{M_{ii}^{\star}} S_d\left(2\pi\sqrt{\frac{M_{ii}^{\star}}{K_{ii}^{\star}}}\right)$$

$\rightarrow$ maximum absolute response in each mode

Response of an M-DOF Oscillator VII



<of course (?) : good agreement between response spectrum & timestepping >

Response of an M-DOF Oscillator VIII

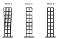
Response in mode i

$$\mathbf{x}_{max}^{(i)} = \phi_i q_{i,max}$$

This displacement field is recovered with the equivalent static forces

$$\mathbf{p}^{(i)} = \mathbf{K}\mathbf{x}_{max}^{(i)} = \mathbf{K}\phi_i \frac{|\mathcal{L}_i|}{M_{ii}^{\star}} S_d(\omega_i;\xi_i)$$

$$= \mathbf{M}\omega_i^2\phi_i \frac{|\mathcal{L}_i|}{M_{ii}^{\star}} S_d(\omega_i;\xi_i) = \mathbf{M}\phi_i \frac{|\mathcal{L}_i|}{M_{ii}^{\star}} S_{pa}(\omega_i;\xi_i)$$



Response of an M-DOF Oscillator IX

Back to displacement and internal forces (combination of modal responses)

Nodal displacements: $\mathbf{x} = \mathbf{\Phi q}$
Internal forces: $\mathbf{F} = (\mathbf{K\Phi})\,\mathbf{q} := \boldsymbol{\varphi}\mathbf{q}$

(i) Combine maximum absolute values (oversafe)

$$x_{k,max} = \sum_{i=1}^{M} \left|\Phi_{ki} q_{i,max}\right| \quad ; \quad F_{k,max} = \sum_{i=1}^{M} \left|\varphi_{ki} q_{i,max}\right|$$

(ii) if modal responses are uncorrelated, use SRSS (square root of the sum of the squares)

$$x_{k,max} = \sqrt{\sum_{i=1}^{M} \left(\Phi_{ki} q_{i,max}\right)^2} \quad ; \quad F_{k,max} = \sqrt{\sum_{i=1}^{M} \left(\varphi_{ki} q_{i,max}\right)^2}$$

Response of an M-DOF Oscillator X

(iii) otherwise, use CQC (complete quadratic combination)

$$x_{k,max} = \sqrt{\sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{ij} \Phi_{kj} \Phi_{ki} q_{i,max} q_{j,max}}$$

$$F_{k,max} = \sqrt{\sum_{i=1}^{M}\sum_{j=1}^{M} \rho_{ij} \varphi_{kj} \varphi_{ki} q_{i,max} q_{j,max}}$$

Further reading I

Méca II, 2017-2018

V. Denoël

Appendix

For further reading

- Clough, R. W. and J. Penzien. *Dynamics of structures.* New-York, McGraw-Hill,1993.
- Géradin, M. and D. Rixen. *Mechanical vibrations: theory and application to structural dynamics, 2002. Ed. Lavoisier.*