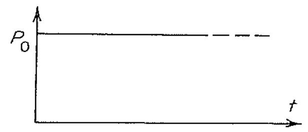
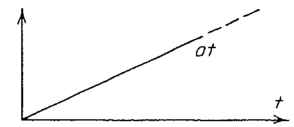
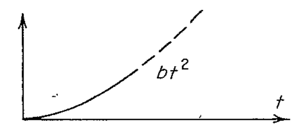
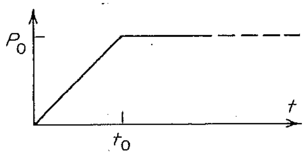
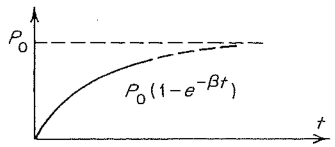
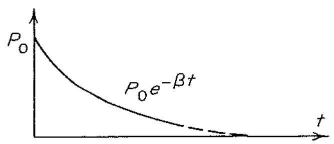
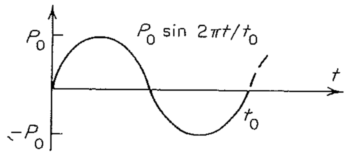
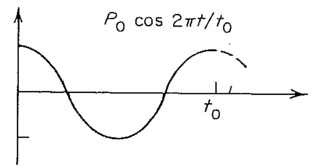
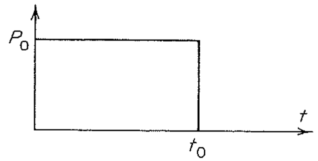
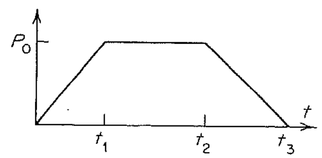
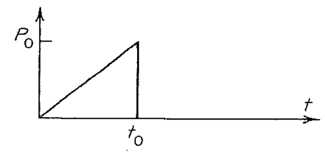
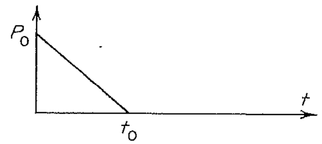
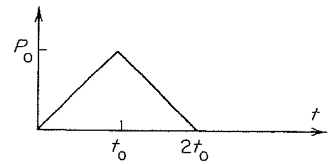
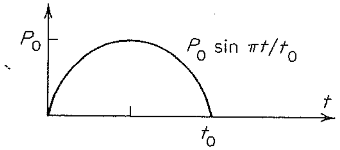
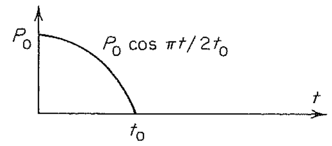
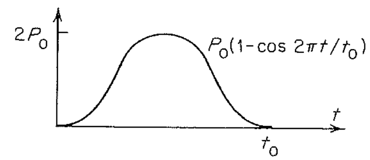
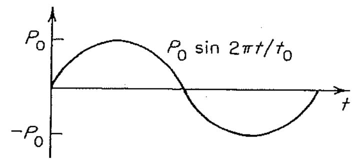
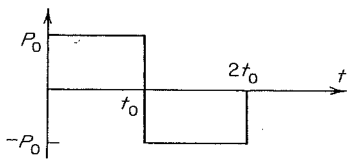
Case no.	Forcing function $\mathscr{P}_e(t)$	$\int_0^t \sin [\omega(t - \tau)]\mathscr{P}_e(\tau)\, d\tau$
1		$\frac{P_0}{\omega}(1 - \cos \omega t)$
2		$\frac{a}{\omega}\left(t - \frac{\sin \omega t}{\omega}\right)$
3		$\frac{b}{\omega}\left(t^2 + \frac{2 \cos \omega t}{\omega^2} - \frac{2}{\omega^2}\right)$
4		$\frac{P_0}{\omega t_0}\left(t - \frac{\sin \omega t}{\omega}\right) \quad t < t_0$ $\frac{P_0}{\omega t_0}\left[t_0 + \frac{\sin \omega(t - t_0)}{\omega} - \frac{\sin \omega t}{\omega}\right] \quad t > t_0$
5		$\frac{P_0}{\omega}(1 - \cos \omega t) + \frac{P_0 \omega}{\omega^2 + \beta^2}\left(-e^{-\beta t} + \cos \omega t - \frac{\beta \sin \omega t}{\omega}\right)$
6		$\frac{P_0 \omega}{\omega^2 + \beta^2}\left(e^{-\beta t} - \cos \omega t + \frac{\beta \sin \omega t}{\omega}\right)$

Case no.	Forcing function $\mathscr{P}_e(t)$	$\int_0^t \sin [\omega(t - \tau)]\mathscr{P}_e(\tau)\, d\tau$
7		$\frac{P_0 t_0}{\omega^2 t_0^2 - 4\pi^2}\left(\omega t_0 \sin 2\pi \frac{t}{t_0} - 2\pi \sin \omega t\right)$
8		$\frac{P_0 \omega t_0^2}{\omega^2 t_0^2 - 4\pi^2} \times \left(\cos 2\pi \frac{t}{t_0} - \cos \omega t\right)$
9		$\frac{P_0}{\omega}(1 - \cos \omega t) \quad t < t_0$ $\frac{P_0}{\omega}[\cos \omega(t - t_0) - \cos \omega t] \quad t > t_0$
10		See case 4 for $t < t_2$; $t_1 = t_0$ $\frac{P_0}{\omega^2 t_1}[\omega t_1 + \sin \omega(t - t_1) - \sin \omega t] - \frac{P_0}{\omega^2(t_3 - t_2)} \times [\omega(t - t_2) - \sin \omega(t - t_2)] \quad t_2 < t < t_3$ $\frac{P_0}{\omega}\left[\frac{\sin \omega(t - t_1)}{\omega t_1} - \frac{\sin \omega t}{\omega t_1} - \frac{\sin \omega(t - t_3)}{\omega(t_3 - t_2)} + \frac{\sin \omega(t - t_2)}{\omega(t_3 - t_2)}\right] \quad t > t_3$

Case no.	Forcing function $\mathscr{P}_e(t)$	$\int_0^t \sin[\omega(t-\tau)]\mathscr{P}_e(\tau)\,d\tau$
11		$\frac{P_0}{\omega t_0}\left(t - \frac{\sin \omega t}{\omega}\right) \quad t < t_0$ $\frac{P_0}{\omega t_0}\left[t_0 \cos \omega(t-t_0) + \frac{\sin \omega(t-t_0)}{\omega} - \frac{\sin \omega t}{\omega}\right] \quad t > t_0$
12		$\frac{P_0}{\omega}\left(1 - \cos \omega t - \frac{t}{t_0} + \frac{\sin \omega t}{\omega t_0}\right) \quad t < t_0$ $\frac{P_0}{\omega}\left[-\cos \omega t - \frac{\sin \omega(t-t_0)}{\omega_0 t_0} + \frac{\sin \omega t}{\omega_0 t_0}\right] \quad t > t_0$
13		$\frac{P_0}{\omega t_0}\left(t - \frac{\sin \omega t}{\omega}\right) \quad t < t_0$ $\frac{P_0}{\omega t_0}\left[2t_0 - t + \frac{2 \sin \omega(t-t_0)}{\omega} - \frac{\sin \omega t}{\omega}\right] \quad t_0 < t < 2t_0$ $\frac{P_0}{\omega^2 t_0}[2 \sin \omega(t-t_0) - \sin \omega(t-2t_0) - \sin \omega t] \quad t > 2t_0$

Case no.	Forcing function $\mathscr{P}_e(t)$	$\int_0^t \sin[\omega(t-\tau)]\mathscr{P}_e(\tau)\,d\tau$
14		$\frac{P_0 t_0}{\omega^2 t_0^2 - \pi^2}\left(\omega t_0 \sin \pi \frac{t}{t_0} - \pi \sin \omega t\right) \quad t < t_0$ $\frac{-P_0 \pi t_0}{\omega^2 t_0^2 - \pi^2}[\sin \omega(t-t_0) + \sin \omega t] \quad t > t_0$
15		$\frac{4P_0 \omega t_0^2}{4\omega^2 t_0^2 - \pi^2}\left(\cos \frac{\pi t}{2t_0} - \cos \omega t\right) \quad t < t_0$ $\frac{-4P_0 t_0^2}{4\omega^2 t_0^2 - \pi^2}\left[\frac{\pi}{2t_0} \sin \omega(t-t_0) + \omega \cos \omega t\right] \quad t > t_0$
16		$\frac{P_0}{\omega}(1 - \cos \omega t) - \frac{P_0 \omega t_0^2}{\omega^2 t_0^2 - 4\pi^2}\left(\cos \frac{2\pi t}{t_0} - \cos \omega t\right) \quad t < t_0$ $\frac{P_0}{\omega}\left\{\cos \omega(t-t_0) - \cos \omega t - \frac{\omega^2 t_0^2}{\omega^2 t_0^2 - 4\pi^2}[\cos \omega(t-t_0) - \cos \omega t]\right\} \quad t > t_0$

Case no.	Forcing function $\mathscr{P}_e(t)$	$\int_0^t \sin[\omega(t-\tau)]\mathscr{P}_e(\tau)\,d\tau$
17		$\frac{P_0 t_0}{\omega^2 t_0^2 - 4\pi^2}\left(\omega t_0 \sin\frac{2\pi t}{t_0} - 2\pi \sin \omega t\right) \quad t < t_0$ $\frac{2\pi P_0 t_0}{\omega^2 t_0^2 - 4\pi^2}[\sin\omega(t-t_0) - \sin\omega t] \quad t > t_0$
18		$\frac{P_0}{\omega}(1-\cos\omega t) \quad t < t_0$ $\frac{P_0}{\omega}[2\cos\omega(t-t_0) - \cos\omega t - 1] \quad t_0 < t < 2t_0$ $\frac{P_0}{\omega}[2\cos\omega(t-t_0) - \cos\omega t - \cos\omega(t-2t_0)] \quad t > 2t_0$

forcing the displacements on the structure to follow the earthquake movements. There are also many other practical applications where the structure is subjected to forced displacements rather than to applied forces.

By using the partitioning in Eq. (13.66) the equations of motion can accordingly be partitioned into

$$\begin{bmatrix} \mathbf{M}_{xx} & \mathbf{M}_{xy} \\ \mathbf{M}_{yx} & \mathbf{M}_{yy} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{U}}_x \\ \ddot{\mathbf{U}}_y \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{xx} & \mathbf{K}_{xy} \\ \mathbf{K}_{yx} & \mathbf{K}_{yy} \end{bmatrix}\begin{bmatrix} \mathbf{U}_x \\ \mathbf{U}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{P}_y \end{bmatrix} \tag{13.67}$$

where $\mathbf{P}_y$ represents the column matrix of unknown forces causing the displacements $\mathbf{U}_y$. Taking the first equation from (13.67), we have

$$\mathbf{M}_{xx}\ddot{\mathbf{U}}_x + \mathbf{M}_{xy}\ddot{\mathbf{U}}_y + \mathbf{K}_{xx}\mathbf{U}_x + \mathbf{K}_{xy}\mathbf{U}_y = \mathbf{0}$$

or

$$\mathbf{M}_{xx}\ddot{\mathbf{U}}_x + \mathbf{K}_{xx}\mathbf{U}_x = \bar{\mathbf{P}}_x \tag{13.68}$$

where

$$\bar{\mathbf{P}}_x = -\mathbf{M}_{xy}\ddot{\mathbf{U}}_y - \mathbf{K}_{xy}\mathbf{U}_y \tag{13.69}$$

The solution to the above modified equation of motion can be obtained in terms of the eigenmodes and frequencies of a constrained system for which $\mathbf{U}_y = \mathbf{0}$. Assuming that $\mathbf{U}_y = \mathbf{0}$ eliminates all the rigid-body degrees of freedom and that $\mathbf{U}_x(0) = \dot{\mathbf{U}}_x(0) = \mathbf{0}$, we obtain from Eq. (13.54)

$$\mathbf{U}_x = \bar{\mathbf{p}}_e \bar{\mathbf{\Omega}}^{-1} \bar{\mathscr{M}}_e^{-1} \int_0^t \mathbf{sin}\,[\bar{\omega}(t-\tau)]\bar{\mathscr{P}}_e(\tau)\,d\tau \tag{13.70}$$

where $\bar{\mathbf{p}}_e$ denotes the eigenmodes of the constrained system and $\bar{\omega}$ the corresponding circular frequencies. Furthermore

$$\bar{\mathbf{\Omega}} = \lceil \bar{\omega}_1 \quad \bar{\omega}_2 \quad \cdots \quad \bar{\omega}_m \rfloor \tag{13.71}$$

$$\bar{\mathscr{M}} = \bar{\mathbf{p}}_e^T \mathbf{M}_{xx} \bar{\mathbf{p}}_e \tag{13.72}$$

$$\bar{\mathscr{P}}_e = \bar{\mathbf{p}}_e^T \bar{\mathbf{P}}_x = -\mathbf{p}_e^T(\mathbf{M}_{xy}\ddot{\mathbf{U}}_y + \mathbf{K}_{xy}\dot{\mathbf{U}}_y) \tag{13.73}$$

Once the displacements $\mathbf{U}_x$ have been calculated, we can compute from Eq. (13.67) the forces $\mathbf{P}_y$ necessary to cause the required displacements $\mathbf{U}_y$.

13.8 DETERMINATION OF FREQUENCIES AND MODES OF UNCONSTRAINED (FREE) STRUCTURES USING EXPERIMENTAL DATA FOR THE CONSTRAINED STRUCTURES

The experimental difficulties of determining mode shapes and frequencies for an unconstrained structure hinge around the problem of supporting the structure during the vibration tests in a manner which will not interfere with the development of free-free modes. This, of course, is very difficult for large structures.

In this section we shall describe an analytical method of determining frequencies and mode shapes for the vibrations of an unconstrained structure using the experimental vibration data for the same structure supported rigidly on the ground. For example, an aircraft can be vibration-tested while supported rigidly at several points on the ground and the experimental results thus obtained can be applied for determining the vibrational characteristics in flight when the structure is not subjected to any external constraints.

We shall assume that the displacements on the unconstrained structure are partitioned into $\mathbf{U}_x$ and $\mathbf{U}_y$. Furthermore, we shall assume that vibration tests to determine frequencies and mode shapes can be performed while supporting the structure in such a manner that $\mathbf{U}_y = \mathbf{0}$ and that all rigid-body degrees of freedom are excluded. As an example, we may use a rocket attached to its launch pad, shown in Fig. 13.4. The displacements $\mathbf{U}_y$ will be those associated with the attachment points, while $\mathbf{U}_x$ will represent all the remaining displacements, the number of which will depend on the idealization of the structure. The equation of motion for a freely vibrating unconstrained system in which $\mathbf{U}_y \neq \mathbf{0}$ is given by

$$\begin{bmatrix} \mathbf{M}_{xx} & \mathbf{M}_{xy} \\ \mathbf{M}_{yx} & \mathbf{M}_{yy} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{U}}_x \\ \ddot{\mathbf{U}}_y \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{xx} & \mathbf{K}_{xy} \\ \mathbf{K}_{yx} & \mathbf{K}_{yy} \end{bmatrix}\begin{bmatrix} \mathbf{U}_x \\ \mathbf{U}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{13.74}$$