Fundamentals of finite elements

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Foreword

This book presents the text of an introductory course on finite element methods. As such, its primary objective is not completeness, but a suitably structured form in a concrete frame. A second aim is to be concept-oriented, not only a compendium of recipes.

In chapter 1, a brief historical sketch is provided, in order to situate the finite element emergence in the frame of the older matrix structural analysis. It is then stated that the finite element method is a particular form of the more classic Ragleigh-Ritz scheme, which is introduced on a very simple problem where it is possible to compare approximate results to an analytic solution.

At this stage, it is useful to develop the variational principles of elasticity, which form a consistent basis for the following. It is the subject of chapter 2, whose deductive exposition follows Fraeijs de Veubeke's views.

It is not possible to develop a constructive theory of finite elements without knowing some general procedures of matrix structural analysis. These procedures may be isolated from any typical finite element problem by considering the simple case of a bar truss. This approach is followed in chapter 3.

Having now these tools in hand, one may enter upon one of the fundamental aspects of the finite element method, that is the field discretization within the element. The second aspect, which is interelement connection, is naturally avoided with beams which are treated in chapter 4. The polynomial expansion of a field may be represented by two ways. The first one, which is the most usual in literature, is based on the shape functions. The second one makes use of a monomial basis and a connection matrix. Although both approaches are in this case equivalent, the second one is of a more general nature and is developed first. But shape functions are also introduced.

A comprehensive discussion of the connection problem, including the possibility of so-called bubble modes and spurious kinematical modes, is given in chapter 5, devoted to plane elasticity. The general exposition is based on the monomial approach, but shape functions are also widely developed. The chapter ends up on the deceipting result that the possible shapes with the polynomal approach are limited to triangles, rectangles and parallelograms only.

The above restriction is a good motivation to go a step further, and introduce in chapter 6 the isoparametric elements. Here appears naturally the necessity of a numerical integration. This procedure is duly discussed, including the problem of spurious kinematical modes that may be implied by a sub-integration.

Plate elements are the subject of chapter 7. It is necessary to first introduce plate theories. The adopted approach is a variational one, and starts with moderate thickness plates. Kirchhoff conditions are set next, and the boundary condition problem with Kirchoff's theory is treated. Turning to finite elements, it is immediately recognized that a conforming thin plate triangular element is not possible with polynomials. The question is thus to find remedies, which are the use of moderate thickness elements, more than conforming elements, or assembled elements. The idea of abandoning strict conformity also directly arise, but at this stage, the corresponding rules are not yet developed.

Chapter 8 is thus an appropriate time to introduce nonconforming elements. After a presentation of old elements of this type, the patch test is introduced as a consistency requirement on equilibrium, which is proved to be equivalent to an incompatibility work. This leads to the zero-interface work version of the patch test. After a verification of this test on the abovementioned elements, a way is shown to specially taylor elements in order to verify the patch test. It is another strategy to circumvent the difficult problem of plates and shells.

The dual analysis concept, one of the older convergence tests, is developed in chapter 9, in its generalized version which works whatever be the boundary conditions. Equilibrium finite elements are not presented in the present text, but its observed that equilibrium approaches may be obtained in the displacement form by using stress functions.

The last chapter briefly exposes the frontal solution algorithm.

As can be seen, this text, as being an introductory course, by no way constitutes a complete treatment of finite element methods. The lacks are too numerous to be listed. It is author's hope to publish a second volume containing complements, including equilibrium, hybrid and mixed models, dynamic analysis, and other useful questions which are beyond the scope of the present lecture notes.

At this stage, it is for me a duty and a pleasure to thank all members of the Aerospace Laboratory who first teached me finite elements and later accepted me as a co-worker. The whole book is strongly influenced by their general philosophy.

I am indebited to Mrs Piffet who patiently typed the text. Finally, truly thanks to my wife for her encouragings.

Contents

CHAPTER 1

INTRODUCTION TO THE FINITE ELEMENT METHOD

1.1. Historical sketch

The finite element method appeared first in the frame of the analysis of aeronautical structures.

1.1.1. - From a long time, structural analysis made use of some hypotheses concerning the diffusion of forces or the kinematics of the deformation of the structural parts. As an example, the classical NAVIER theory of beams is based on the following assumptions (fig. 1).

a) The stress state is unidimensional
b) Orthogonal sections remain plane during deformation
c) Furthermore, they remain orthogonal to the neutral axis.

The first assumption leads to the simple St form of Hooke's law

$$\sigma = E\varepsilon \tag{1.1}$$

The second one implies that the u- displacement varies linearly in terms of y.

$$u(x,y) = \alpha\,(x)y \tag{1.2}$$

From condition c, the value of α is related to the angular deflection of the beam $\frac{dv}{dx}$ by the condition

$$\alpha + \frac{\partial v}{\partial x} = 0 \tag{1.3}$$

(Remember that we are here confined in the frame of the so called small displacements or more properly speaking, of small strains and small rotations). The solution obtained by this theory does generally not coincide with the exact solution of the elasticity equations, but for sufficiently long beams, the difference is small, if one excepts some narrow zones near supports and loads (Saint-Venant's principle).

1.1.2. - Another celebrated case of approximation is the articulated truss. Physically speaking, the truss is componed from beams, it is to say that each truss component is able to resist to some flexure. The articulated truss idealization consists to consider that each bar only works in stretching and that it is articulated at its extremities (fig. 2). If then g is the force acting on the bar and q_1 and q_2 the displacements of extremities 1 and 2 respectively (fig. 3), the bar obeys to the relation

$$g = k(q_1 - q_2) \tag{1.4}$$

where k is the bar stiffness. The connections of bars are called *nodes*. At each node, the displacement of each bar may be decomposed in a structural frame (fig. 4) as

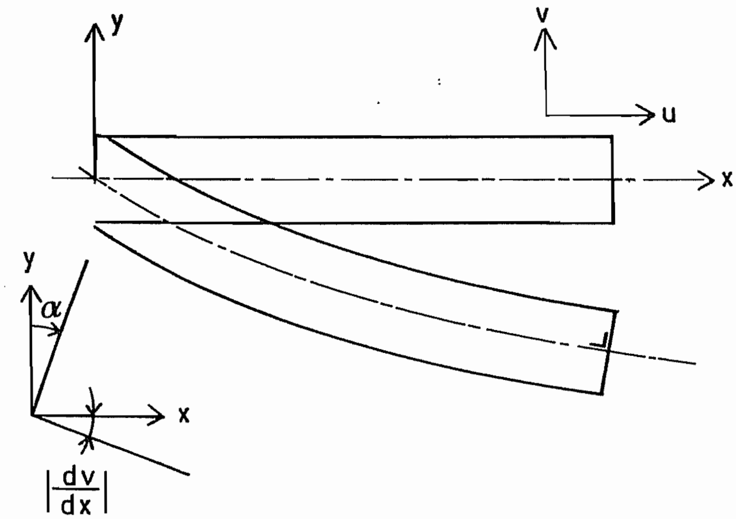


Fig. 1

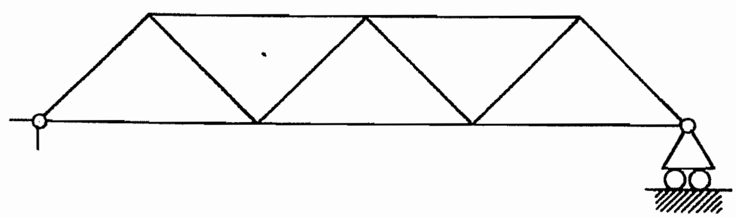


Fig. 2

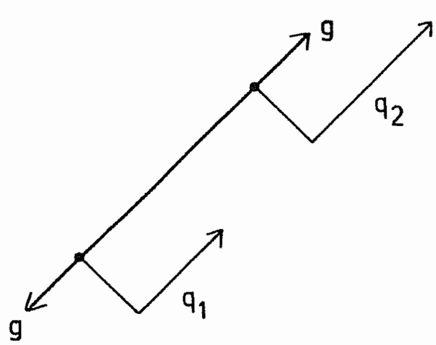


Fig. 3

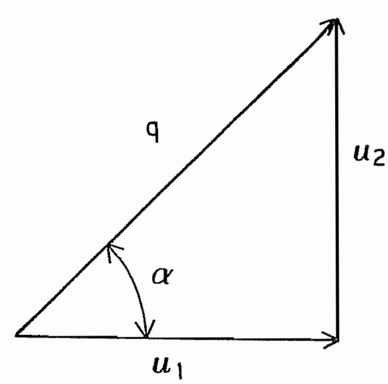


Fig. 4

$$q = u\cos\alpha + v\sin\alpha \tag{1.5}$$

and the force has two components

$$X = g\cos\alpha, \qquad Y = g\sin\alpha \tag{1.6}$$

Now, assembling the different bars may be performed by two ways :

(i) Express that the displacement of each node is uniquely determined and then find the value of the nodal displacements which leads to equilibrium. This is the *displacement method.*

(ii) Express that the sum of forces at each node is zero (or eventually, equilibrates the applied load). The solution of these equations is undetermined, depending on arbitrary self-stresses whose number is equal to the hyperstaticity index. The self-stresses are then adjusted to ensure the uniqueness of the displacement at each node. This is the *force method.*

Both methods lead to a matrix system to be solved.

1.1.3. - Aeronautical structures largely use panels and stiffeners. A classical approximation consists to consider that a panel only rèsists to shear, an that this shear stress is constant along the panel (in fact, this is the case if buckling of the panel occurs) (fig. 5). Stretching is then supported by the frames. Associating displacements to the shear forces, one is also leaded to a matrix system to solve.

1.1.4. - All these idealizations are parts of the so-called *matrix structural analysis*. The first complete treatment of this subject is due to ARGYRIS (1954) [1]. At this stage, it may be considered as an exact solution of a structure which is *idealized on a physical basis*. In fact, it is true that a structure is really composed of frames, bars, panels. And in each element, use is made of classical hypotheses which permit to evaluate their stiffness with confidence. Matrix structural analysis is thus only a systematized way of solving a structure.

1.1.5. - In 1956, TURNER, CLOUGH, MARTIN and TOPP [2], working on thin walled structures, thought of a new procedure. Subdividing *arbitrarily* each wall in triangular elements (fig. 6), they supposed that in each triangle, the displacement is linear, say

$$\left.\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned}\right\} \tag{1.7}$$

It is then possible to express the α_i's in terms of the displacements of the corners (nodal displacements). Furthermore, in each element, stresses and strains are constant, in such a way that is is easy to reckon the force corresponding to any nodal displacement. Finally, connecting the nodal displacements ensures compatibility on each interface.

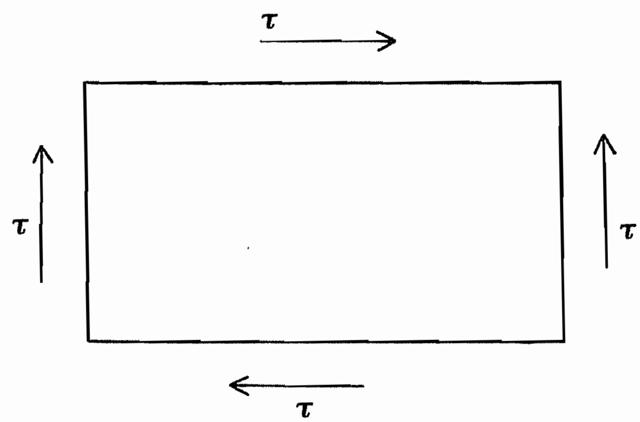


Fig. 5

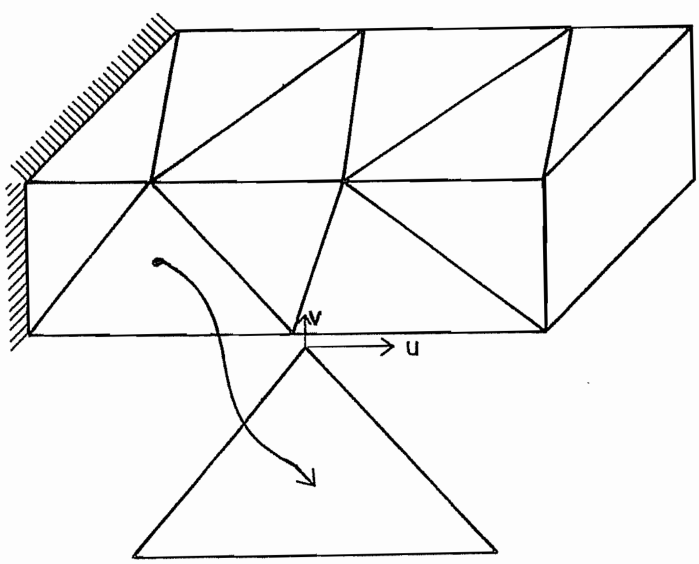


Fig. 6

This was the starting point of finite element methods. This idea that, following oral tradition, the authors hesistated to publish, judging it of trivial nature, had such resonance in engineer's world that in 1959, a first finite element congress took place in Aachen (Germany). However, some reluctance emerged in scientific milieus, from the fact that finite elements seemed not to have any justification.

1.1.6. - A further step was the recognition of the fact that the finite element method is nothing than a particular form of the well-known RAYLEIGH-RITZ procedure. As is well known, Rayleigh-Ritz procedure consists to define a basis of functions and to seek the coefficients of these functions which minimize a given functional. In fact, finite elements lead to a particularly appropriated basis.

1.1.7. - From this time, large progresses have been made in finite element techniques and in their analysis, which proceeds from functional analysis (SOBOLEV Spaces). In this course, we will restrict ourselves to some techniques whose choice is by some extend arbitrary, but which cover essential needs of structural engineers.

1.2. An introductory problem

1.2.1. - To illustrate the different ideas which conduct to finite elements, let us consider the very simple problem of a string taut by a force N. If p(x) is the transversal load by unit length, the differential equation of the displacement is (fig. 7)

$$-N\frac{d^2u}{dx^2} = p(x) \tag{1.8}$$

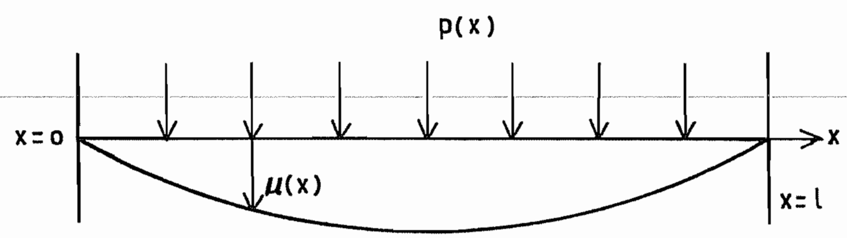


Fig. 7

with the boundary conditions u(o) = 0, u(l) = 0. The first step toward an approximate solutions is to find a *variational principle* which is equivalent to equation (1.8) with the given boundary conditions. For this, let us first note that every candidate for the solution u is equal to zero at both ends. Such a displacement field will be called a *kinematically admissible displacement*. A *kinematically admissible displacement variation* δu is then defined as an arbitrary difference between two admissible displacements. It is therefore equal to zero at both ends. Now, multiply equation (1.8), written as

$$-N\frac{d^2u}{dx^2}-p(x)=0$$

by such a displacement variation, and integrate from 0 to l. One obtains

$$0=\int_o^l\left(-N\frac{d^2u}{dx^2}\delta u-p(x)\delta u\right)dx\,.$$

Integrating the first term by parts leads to

$$-\int_o^l N\frac{d^2u}{dx^2}\delta u\,dx=-\left[N\frac{d^2u}{dx^2}\delta u\right]_o^l+\int_o^l N\frac{du}{dx}\frac{d\delta u}{dx}dx$$

From the fact that δu(o) = δu(l)=o, this reduces to

$$\int_o^l N\frac{du}{dx}\frac{d\delta u}{dx}dx=\frac{1}{2}\delta\int_o^l N\left(\frac{du}{dx}\right)^2dx$$

Therefore we obtain

$$\delta\int_o^l\left[\frac{1}{2}N\left(\frac{du}{dx}\right)^2-pu\right]dx=0$$

or

$$\delta\mathscr{F}(u)=0 \tag{1.9}$$

with

$$\mathscr{F}(u)=\frac{1}{2}\int_o^l N\left(\frac{du}{dx}\right)^2dx-\int_o^l pu\,dx \tag{1.10}$$

This is to say that the solution u of problem (1.8) renders the functional $\mathscr{F}(u)$ stationary.

1.2.2. - The classical RAYLEIGH-RITZ method consists in seeking, among all superpositions of given functions u, the combination that renders (1.10) stationary. Let us, as an example, develop u in the form

$$u = \sum_{k=1}^{m} A_k \sin \frac{k\pi x}{l} \tag{1.11}$$

and suppose, for simplicity, p(x) = p = ct. One has

$$\mathcal{F}(u) = \frac{1}{2}\sum_{k=1}^{m} N A_k^2 \frac{k^2\pi^2}{l^2} - p\sum_{k=1}^{m} \frac{l}{k\pi}\left[1-(-1)^k\right] A_k = \mathcal{F}(A_1,...A_m)$$

To find the stationary point, we have only to derive $\mathcal{F}$ by respect of A_1, ..., A_m. One obtains

$$\frac{\partial \mathcal{F}}{\partial A_k} = N A_k \frac{k^2\pi^2}{2l} - p\frac{l}{k\pi}\left[1-(-1)^k\right] = 0,$$

so that

$$A_k = \begin{cases} \dfrac{4l^2}{k^3\pi^3 N} p & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \tag{1.12}$$

The displacement at x = l/2 is then

$$u\left(\frac{l}{2}\right) = A_1 - A_3 + A_5 - A_7 ... = \frac{4pl^2}{\pi^3 N}\left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} ...\right] \tag{1.13}$$

This gives the following table :

m	$\dfrac{N\pi^3 u(l/2)}{4pl^2}$
1	1
3	0,9630
5	0,9710
7	0,9680
9	0,9694
11	0,9687
13	0,9691
15	0,9688

The *exact* solution is obtained from (1.8) :

$$\frac{d^2u}{dx^2} = -\frac{p}{N}$$

$$\frac{du}{dx} = -\frac{px}{N} + C$$

$$u = -\frac{px^2}{2N} + Cx + D$$

and

$$u(o) = D = 0$$
$$u(l) = -\frac{pl^2}{2N} + Cl = 0,$$

from which

$$C = +\frac{pl}{2N}$$

and

$$u = \frac{p}{2N}(-x^2 + lx)$$

So,

$$u(l/2) = \frac{pl^2}{8N} = \frac{4pl^2}{\pi^3 N} \cdot \frac{\pi^3}{32} = 0{,}9689 \cdot \frac{4pl^2}{\pi^3 N} \tag{1.14}$$

As can be seen, the displacement at x = l/2 converges to the true value as $m \to \infty$. More interesting is the work of the load p,

$$\mathcal{T} = \int_o^l p\ u\ dx$$

Its true value is

$$\mathcal{T} = \frac{p^2}{2N}\int_o^l (lx - x^2)dx = \frac{p^2 l^3}{12N} \tag{1.15}$$

The approximate values, from the above Rayleigh-Ritz process, are

$$\mathcal{T}_m = \frac{8p^2 l^3}{\pi^4 N} \cdot \sum_{\substack{k=1 \\ odd}}^{m} \frac{1}{k^4} = \frac{p^2 l^3}{12N} \cdot \frac{96}{\pi^4} \sum_{\substack{k=1 \\ odd}}^{m} \frac{1}{k^4} \tag{1.16}$$

as indicated in the following table

m	1	3	5	7	9
$\frac{12N\mathcal{T}_m}{p^2l^3}$	0,9855	0,9977	0,9993	0,9997	0,9998

from which it is clear that a very fair convergence exists. We will prove later that this is a very general result.

1.2.3. - The finite element method consists to cut the interval]o,l[in subintervals $]x_i, x_{i+1}[$ (i = o, ..., n-1), where it is posed

$$u(x) = u_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + u_{i+1} \frac{x - x_i}{x_{i+1} - x_i} \tag{1.17}$$

(linear interpolation between nodal values u_i and u_{i+1}, see fig. 8).

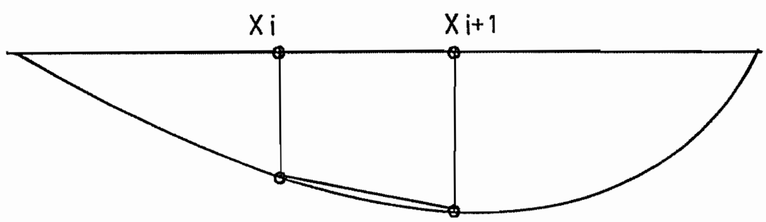


Fig. 8

One has thus

$$\frac{du}{dx} = \frac{u_{i+1} - u_i}{x_{i+1} - x_i} \tag{1.18}$$

In each subinterval, which is called an *element*, one calculates

$$\frac{1}{2}\int_{x_i}^{x_{i+1}} N\left(\frac{du}{dx}\right)^2 dx = \frac{1}{2}\int_{x_i}^{x_{i+1}} N\left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i}\right)^2 dx$$

and

$$\int_{x_i}^{x_{i+1}} pu\,dx = u_i \int_{x_i}^{x_{i+1}} p\frac{x - x_{i+1}}{x_i - x_{i+1}}\,dx + u_{i+1}\int_{x_i}^{x_{i+1}} p\frac{x - x_i}{x_{i+1} - x_i}\,dx$$

Summing on all elements, one obtains a function of n variables

$$\mathscr{F}(u_1,\ldots,u_n) = \sum_{i=o}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{1}{2}N\left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i}\right)^2 - p\left(u_i\frac{x - x_{i+1}}{x_i - x_{i+1}} + u_{i+1}\frac{x - x_i}{x_{i+1} - x_i}\right)\right]dx$$

which can be minimized.

Note that it is a Rayleigh-Ritz method where the basis functions are of the roof-type (fig. 9),

$$u(x) = \sum_{i=1}^{n-1} M_i(x)\,u_i$$

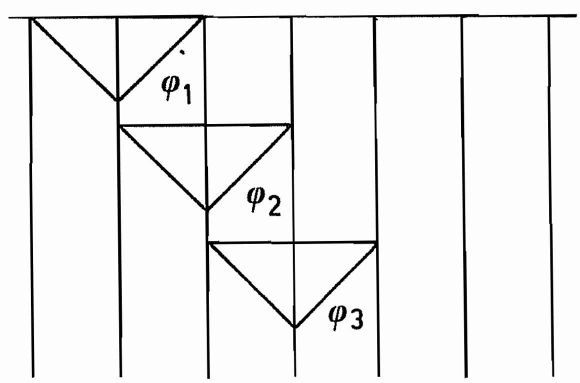


Fig. 9

with

$$M_i(x) = \begin{cases} 0 & \text{if } x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} < x \le x_i \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & \text{if } x_i < x \le x_{i+1} \\ 0 & \text{if } x > x_{i+1} \end{cases}$$

It may be proved that the nodal values are exact in this approach [3], but this is a special property of one-dimensional finite elements.

Note finally that for the piecewise linear approximation which is used, the original differential equation has no sense. The variational principle, however, is well defined. This is a specificity of finite elements : the approximate solutions are just able to ensure the existency of the variational principle, not of the local differential equation. In this direction, note that not regular functions are not unusual in engineering practice. As an example, if the string is submitted to a concentrated load (fig. 10), the solution is precisely of the roof-type.

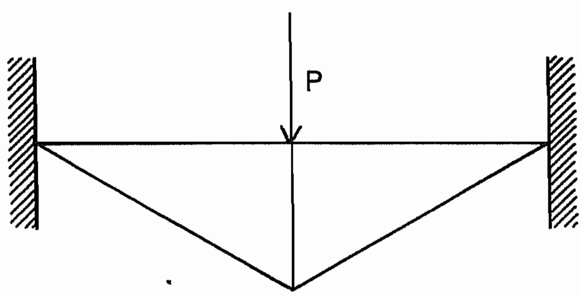


Fig. 10

CHAPTER 2

VARIATIONAL PRINCIPLES IN ELASTICITY

2.1. Introduction

Variational principles are the key of developing approximate solutions of elasticity. They form also the basis of Rayleigh-Ritz approximations, including finite element methods.

Let us consider an elastic body whose volume will be denoted V (fig. 11). Its boundary may be splitted in two parts S_1 and S_2, with prescribed displacements $\bar{u}_i$ on S_1 and imposed surface tractions $\bar{t}_i dS$ on S_2. Moreover, volumic forces $f_i dV$ are applied inside the body. In what follows, it is always assumed that fixations are sufficient to ensure displacement unicity.

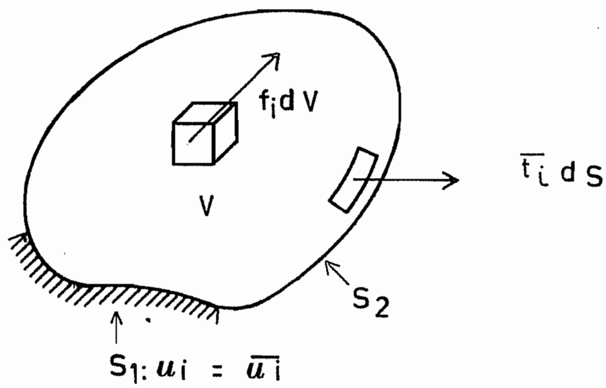


Fig. 11

The general equations between the stress field σ, the srain field ε, the displacement **u**, field, and the data are as follows.

- <u>Compatibility equations</u>

$$\begin{cases} \varepsilon_{ij} = \frac{1}{2}(D_i u_j + D_j u_i) \; in \; V \\ u_i = \bar{u}_i \; on \; S_1 \end{cases} \tag{2.1}$$

- <u>Constitutive relations</u> : it is assumed that the body is *hyperelastic*, that is, there exists a function W, called the *strain energy density*, such that

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{2.2}$$

and that for each field $\delta \varepsilon_{ij} \neq 0$,

$$\frac{\delta^2 W}{\partial \varepsilon_{ij} \, \partial \varepsilon_{kl}} \delta \varepsilon_{ij} \, \delta \varepsilon_{kl} > 0 \tag{2.3}$$

In the linear case, one has simply

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \tag{2.4}$$

where C_{ijkl} is Hooke's tensor, and condition (2.3) reduces to

$$C_{ijkl} \, \delta\varepsilon_{ij} \, \delta\varepsilon_{kl} > 0 \tag{2.5}$$

- Equilibrium equations

$$\begin{cases} D_j \sigma_{ji} + \bar{f}_i = 0 \; in \, V \\ n_j \sigma_{ji} = \bar{t}_i \; on \; S_2 \end{cases} \tag{2.6}$$

Each variational principle will be equivalent to some of these equations, the other ones having to be verified *a priori*.

2.2. Minimum total energy principle

In this principle, compatibility equations and constitutive relations are verified *a priori*. An *admissible displacement field* u is by definition a displacement field verifying.

$$\begin{cases} \int_V W(Du) \, dx < \infty \\ u_i = \bar{u}_i \; on \; S_1 \end{cases} \tag{2.7}$$

where the notaion W(Du) is used to recall that the strain energy W(ε) is computed from displacements. Now, an *admissible displacement variation* δu is the difference between two admissible displacements. Consequently, it verifies

$$\delta u_i = u_i^{(1)} - u_i^{(2)} = \bar{u}_i - \bar{u}_i = 0 \quad on \; S_1 \tag{2.8}$$

This being admitted, let us multiply equations (2.6) by an admissible displacement variation δu_i. This gives

$$\delta u_i (D_j \sigma_{ji} + \bar{f}_i) = 0 \quad in \; V$$
$$\delta u_i (n_j \sigma_{ji} - \bar{t}_i) = 0 \quad on \; S_2 .$$

Integrate the first relation on the volume, and the second on S_2, and make the difference. One obtains

$$-\int_V \delta u_i D_j \sigma_{ji} dV - \int_V \bar{f}_i \delta u_i dV + \int_{S_2} n_j \sigma_{ji} \delta u_i dS - \int_{S_2} \bar{t}_i \delta u_i dS = 0 \qquad (2.9)$$

The first term of this equation may be transformed as follows. First, an integration by parts leads to

$$-\int_V \delta u_i D_j \sigma_{ji} \, dV = -\int_S n_j \sigma_{ji} \delta u_i \, dS + \int_V \sigma_{ji} D_j \delta u_i \, dV$$

Now, on the surface, $\delta u_i \neq 0$ only on S_2, and the term on S is balanced by the third term of 2.9. It remains the term on the volume. In this one, note that $\sigma_{ij} = \sigma_{ji} = \frac{1}{2}(\sigma_{ij} + \sigma_{ji})$

$$\int_V \sigma_{ji} D_j \delta u_i \, dV = \frac{1}{2}\int_V (\sigma_{ij} + \sigma_{ji}) \, D_j \delta u_i dV$$

Here, indexes i and j are summation indexes and may therefore be replaced by other ones. So,

$\sigma_{ij} D_j \delta u_i = \sigma_{kl} D_l \delta u_k$
$\sigma_{ji} D_j \delta_{ui} = \sigma_{kl} D_k \delta u_l$

and

$$\frac{1}{2}(\sigma_{ij} + \sigma_{ji}) D_j \delta u_i = \sigma_{kl} \frac{1}{2}(D_k \delta u_l + D_l \delta u_k) = \sigma_{kl} \delta \varepsilon_{kl} = \delta W,$$

where use is made of the compatibility equations and the constitutive relations.

Finally, (2.9) transforms in

$$\delta \left\{ \int_V \left[W(Du) - \bar{f}_i u_i \right] dV - \int_{S_2} \bar{t}_i u_i dS \right\} = 0 \qquad (2.10)$$

or

$$\delta \mathscr{E} = \delta(\mathscr{U} + \mathscr{P}) = 0 \qquad (2.11)$$

where

$$\mathscr{U} = \int_V W(Du) \, dV \qquad (2.12)$$

is the *strain energy*

$$\mathcal{P} = -\int_V \bar{f}_i u_i dV - \int_{S_2} \bar{t}_i u_i dS \tag{2.13}$$

is the *potential energy of the loads*, and

$$\mathcal{E} = \mathcal{U} + \mathcal{P} \tag{2.14}$$

is the *total energy*. We have thus established that at the equilibrium, the total energy is *stationary*. The fact that this stationary point is a ***minimum*** results from the positive definiteness of the second variation

$$\delta^2\mathcal{E} = \int_V \frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} \delta\varepsilon_{ij}\delta\varepsilon_{kl} dV > 0$$

which is a consequence of assumption (2.3).

2.3. Equivalence of the minimum total energy principle to equilibrium

Up to now, we have proved that equilibrium implies the stationarity of the total energy. But is the converse true ? Let us suppose that the total energy is stationary. This is to say that for every admissible displacement variation, $\delta\mathcal{E} = 0$. But

$$\begin{aligned}\delta\mathcal{E} &= \int_V \sigma_{ij} D_j \delta u_i dV - \int_V \bar{f}_i \delta u_i dV - \int_{S_2} \bar{t}_i \delta u_i dS \\ &= \int_S \sigma_{ij} n_j \delta u_i dS - \int_V \left(D_j \sigma_{ji} + \bar{f}_i\right) \delta u_i dV - \int_{S_2} \bar{t}_i \delta u_i dS\end{aligned}$$

and, owing to the fact that $\delta u_i = 0$ on S_1,

$$\delta\mathcal{E} = \int_{S_2} \left(\sigma_{ij} n_j - \bar{t}_i\right) \delta u_i dS - \int_V \left(D_j \sigma_{ji} + \bar{f}_i\right) \delta u_i dV = 0 \tag{2.15}$$

Here, a particular result of integration theory is needed. This result is as follows. *Let $\mathcal{D}(V)$ be the space of functions φ that are indefinitely continuously differentiable, and whose support (i.e. the set where $\varphi \neq 0$) is a compact included in the open set V* (It is to say that $\varphi = 0$ in the neighborhood of the boundary*). Then, if g is a locally integrable function such that for each $\varphi \in \mathcal{D}(V)$,*

$$\int_V g\varphi \, dV = 0$$

one can deduce that g = 0 almost everywhere in V.

Let us consider the subset of admissible displacement variations $\delta\varphi_i$ such that $\delta\varphi_i \in \mathscr{D}(V)$. For such variations,

$$\delta\mathscr{E} = -\int_V \left(D_j\sigma_{ji} + \bar{f}_i\right)\delta\varphi_i dV = 0$$

(because $\delta\varphi_i = 0$ on S), and, from the citated theorem, this implies

$$D_j\sigma_{ji} + \bar{f}_i = 0 \text{ a.e. in V} \tag{2.16}$$

i.e., internal equilibrium. This result being known, it is clear that for any variation δu_i,

$$\int_V \left(D_j\sigma_{ji} + \bar{f}_i\right)\delta u_i \, dV = \int_V O.\delta u_i \, dV = 0,$$

so that (2.15) reduces to

$$\int_{S_2} \left(\sigma_{ij}n_j - \bar{t}_i\right)\delta u_i dS = 0$$

and, from the arbitrariness of δu_i, this implies

$$\sigma_{ij}n_j = \bar{t}_i \text{ on } S_2 \tag{2.17}$$

The equivalence is thus proved.

However, this equivalence holds only if the stresses are sufficiently smooth so that local equilibrium equations can have a meaning. In particular, $D_j\sigma_{ji}$ must exist to make sense at these equations. In contrary, $\delta\mathscr{E}$ exists whenever σ_{ij} is square-integrable, a condition which is by far less strong. In fact, *the variational formulation is more general than the local one*. In mathematical theories, the expression $\delta\mathscr{E} = 0$ is called the *weak form* of equilibrium equations.

2.4. The general principle of elasticity

In the minimum total energy principle, compatibility and constitutive equations are satisfied *a priori*. One says that they are *essential conditions*. The result of the variation, also called *natural conditions*, is the equilibrium. Is is possible to obtain a general principle where all conditions are natural. To do this, we start with the total energy where we suppress the a priori condition that $\varepsilon = \frac{1}{2}\left(D_i u_j + D_j u_i\right)$,

$$\mathscr{E}(\varepsilon,u) = \int_V W(\varepsilon)\, dV - \int_V \bar{f}_i u_i dV - \int_{S_2} \bar{t}_i u_i dS \tag{2.18}$$

We then add [6] *dislocation potentials* of the form (Lagrange-multiplier x condition)

$$\mathcal{Q}_1(\lambda,u,\varepsilon)=\int_V \lambda_{ij}\left\{\frac{1}{2}(D_iu_j+D_ju_i)-\varepsilon_{ij}\right\}dV \tag{2.19}$$

for the internal compatibility, and

$$\mathcal{Q}_2(\mu,u)=\int_{S_1}\mu_i(\overline{u}_i-u_i)\,dS \tag{2.20}$$

to relax the kinematical conditions on S_1. Now, we express that

$$\delta\mathcal{H}=\delta(\mathcal{E}+\mathcal{Q}_1+\mathcal{Q}_2)=0 \tag{2.21}$$

where variations are taken on u, ε,λ,μ, *without any restriction*. This principe is generally associated with the names of Hu and Washizu, who published it independently in 1955. It has however to be pointed that this functional had been derived and published four years earlier by Fraeijs de Veubeke [52]. Let us examine the responsabilities of each variation.

a) Varying the strains leads to the condition

$$\int_V\left(\frac{\partial W}{\partial\varepsilon_{ij}}-\lambda_{ij}\right)\delta\varepsilon_{ij}dV=0$$

and, due to the arbitrariness of δε,

$$\frac{\partial W}{\partial\varepsilon_{ij}}=\lambda_{ij}\ \text{in V}$$

The field λ may thus be interpreted as energetical stresses, i.e. stresses that are obtained from the *constitutive equations.*

b) Varying λ_{ij} leads to the conditions

$$\frac{1}{2}\left(D_iu_j+D_ju_i\right)-\varepsilon_{ij}=0\ \text{in V},$$

which express *internal compatibility*

c) Varying μ_i gives the S_1 *–compatibility.*

$$u_i=\overline{u}_i\ \text{on S}_1$$

d) Varying the displacements leads, after an integration by parts,

$$-\int_V\left(D_j\lambda_{ji}+f_i\right)\delta u_idV+\int_{S_2}\left(n_j\lambda_{ji}-\overline{t}_i\right)\delta u_idS+\int_{S_1}\left(n_j\lambda_{ji}-\mu_i\right)\delta u_idS$$

and, by the same way as in section (2.3), this gives

$$D_j\lambda_{ji} + f_i = 0 \quad in\, V$$
$$n_j\lambda_{ji} = \bar{t}_i \qquad on\ S_2$$
$$n_j\lambda_{ji} = \mu_i \qquad on\ S_1$$

The two first equations express equilibrium, λ_{ij} being interpreted as stresses. The last one permits to interpret the surface multiplicators μ_i as the reactions on S_1.

As can be seen, all elasticity equations are contained in this general principle which theoretically could serve as a basis for complex approximations. However, it is seldom used for two reasons. Firstly, one can hardly conceive approximations on the constitutive equations (except, perhaps, numerical integration). Secondly, simultaneous restrictions on several fields require consistency conditions which are eventually difficult to express. Practical applications are thus developed from less general principles that we will obtain by assuming some *a priori* conditions.

2.5. HELLINGER-REISSNER's principle

This principle is obtained by imposing in the general principle an a priori verification of constitutive equations.

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij}$$

where we use the notation σ_{ij} in place of λ_{ij} for the arbitrary stresses, as is traditional. The two-field function

$$A\,(\sigma, \varepsilon) = \sigma_{ij}\varepsilon_{ij} - W\,(\varepsilon) \tag{2.22}$$

which appears in the principle has the following properties :

$$\frac{\partial A}{\partial \sigma_{ij}} = \varepsilon_{ij} \text{ whatever be } \varepsilon$$

$$\frac{\partial A}{\partial \varepsilon_{ij}} = \sigma_{ij} - \frac{\partial W}{\partial \varepsilon_{ij}}$$

Let us define the function

$$\Phi(\sigma) = \max_{\varepsilon} A(\sigma,\varepsilon) \tag{2.23}$$

This is a function of the stresses only, whose derivatives are :

$$\frac{\partial \Phi}{\partial \sigma_{ij}} = \frac{\partial A}{\partial \sigma_{ij}} = \varepsilon_{ij} \tag{2.24}$$

In the case of linear constitutive equations,

$$W(\varepsilon) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \tag{2.25}$$

and

$$A(\sigma, \varepsilon) = \sigma_{ij} \varepsilon_{ij} - \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$$

To obtain $\Phi(\sigma)$, one eliminates ε from the maximum condition

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

which leads

$$\varepsilon_{ij} = C^{-1}_{ijkl} \sigma_{kl}$$

The result is

$$\Phi(\sigma) = \frac{1}{2} C^{-1}_{ijkl} \sigma_{ij} \sigma_{kl} \tag{2.26}$$

We thus take the maximum of the general functional $\mathscr{H}$ by respect to ε, and obtain the functional

$$\mathscr{R}(u,\sigma,\mu) = \int_V \left(\sigma_{ij} \frac{1}{2}\left(D_i u_j + D_j u_i\right) - \Phi(\sigma) - f_i u_i \right) dV - \int_{S_2} \bar{t}_i u_i dS + \int_{S_1} \mu_i (\bar{u}_i - u_i)\, dS \tag{2.27}$$

whose stationarity express HELLINGER-REISSNER's principle [5]. In this principle, the variation of stresses is responsible of the compatibility equations in the form

$$\frac{1}{2}(D_i u_j + D_j u_i) = \frac{\partial \Phi}{\partial \sigma_{ij}} (= \varepsilon_{ij})$$

which combines them with stress-strain relations. Varying μ_i, one obtains as precedently the S_1 – compatibility

$$u_i = \bar{u}_i$$

Finally, the variation of displacements leads to the same relations as in the general principle, it is equilibrium.

Hellinger-Reissner principle is widely used in theoretical approximations and also form the basis of so-called mixed finite elements.

2.6. FRAEIJS de VEUBEKE's principle

Another principle may be obtained from the general principle by a priori imposing *equilibrium*, in its weak form, i.e.

$$\int_V \sigma_{ij} \frac{1}{2}\left(D_i \delta u_j + D_j \delta u_i\right) dV - \int_V \bar{f}_i \delta u_i dV - \int_{S_2} \bar{t}_i \delta u_i dS - \int_{S_1} \mu_i \delta u_i dS = 0$$

Noting that here, no restriction is made on the displacements on S_1, displacements themselves are admissible displacement variations, so that

$$\int_V \sigma_{ij} \frac{1}{2}(D_i u_j + D_j u_i)\, dV - \int_V f_i u_i dV - \int_{S_2} \bar{t}_i u_i dS - \int_{S_1} \mu_i u_i dS = 0$$

Moreover, equilibrium implies on S_1

$$\mu_i = n_j \sigma_{ji}$$

This all leads to the following principle

$$\delta \mathcal{F}(\varepsilon,\sigma) = 0 \text{ (}\varepsilon\text{ arbitrary, }\sigma\text{ in equilibrium)} \qquad (2.27b)$$

with

$$\mathcal{F}(\varepsilon,\sigma) = \int_V (W(\varepsilon) - \sigma_{ij}\varepsilon_{ij})\, dV + \int_{S_1} n_j \sigma_{ji} \bar{u}_i dS \qquad (2.28)$$

This principle is due to FRAEIJS de VEUBEKE [6]. The variation of the strains gives the constitutive equations

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij}\,.$$

Stresses may not be varied independently, so that it is not possible to derive here a local equation. The only condition that may be obtained is

$$-\int_V \varepsilon_{ij} \delta\sigma_{ij} dV + \int_{S_1} n_j \delta\sigma_{ji} \bar{u}_i dS = 0 \qquad (2.29)$$

<u>for every self-stress field</u>. In fact, equilibrated stresses verify

$$\left.\begin{aligned} D_j\sigma_{ji} &= 0 \; in \; V \\ n_j\sigma_{ji} &= \bar{t}_i \; on \; S_2 \end{aligned}\right\} \quad (2.30)$$

An admissible stress variation will then be the difference of two equilibrated stress fields,

$$\delta\,\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$$

each of these verifying (2.30). Consequently,

$$\begin{cases} D_j\delta\,\sigma_{ji} = 0 \; in \; V \\ n_j\delta\,\sigma_{ji} = 0 \; on \; S_2 \end{cases} \quad (2.31)$$

in other words, $\delta\sigma_{ij}$ is a self-stress field. Condition (2.29) is the weak form of compatibility (internal and on S_1).

2.7. Complementary energy principle

If we now eliminate ε from FRAEIJS de VEUBEKE's principle by a priori taking the minimum of $\mathscr{F}(\varepsilon, \sigma)$ by respect of ε, one obtains

$$\begin{aligned} \min_{\varepsilon}\mathscr{F}(\varepsilon,\sigma) &= \int_V \min_{\varepsilon}\left(W(\varepsilon) - \sigma_{ij}\varepsilon_{ij}\right) dV + \int_{S_1} n_j\sigma_{ji}\bar{u}_i dS \\ &= -\int_V \Phi(\sigma)\, dV + \int_{S_1} n_j\sigma_{ji}\bar{u}_i dS \end{aligned}$$

Changing the sign, one is leaded to

$$\delta\mathscr{C}(\sigma) = 0 \quad (2.32)$$

where

$$\mathscr{C}(\sigma) = \mathscr{V}(\sigma) + \mathscr{D}(\sigma) \quad (2.33)$$

is the *total complementary energy* composed of the *complementary strain energy*

$$\mathscr{V}(\sigma) = \int_V \Phi(\sigma) dV \quad (2.34)$$

and the *potential of prescribed displacements*

$$\mathscr{D}(\sigma) = -\int_{S_1} n_j\sigma_{ji}\bar{u}_i dS \quad (2.35)$$

The principle contains the compatibility conditions written in the form

$$\int_V \frac{\partial \Phi}{\partial \sigma_{ij}} \delta\sigma_{ij} dV - \int_{S_1} n_j \delta\sigma_{ji} \bar{u}_i dS = 0 \tag{2.36}$$

for every self-stress field $\delta\sigma_{ij}$

2.8. Note on the weak compatibility condition

In what follows, we restrict ourselves to the linear case. Let us note E the space of square integrable stresses σ_{ij}, with the scalar product

$$(\sigma, \tau) = \int_V C^{-1}_{ijkl} \sigma_{ij} \tau_{kl} \, dV \tag{2.37}$$

A compatible stress field will be defined as a stress field of the form

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}(u) \tag{2.38}$$

where u is an admissible displacement field, i.e.

$$\int_V C_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(u) dV < \infty \tag{2.39}$$

$$u_i = \bar{u}_i \ on \ S_1 \tag{2.40}$$

A *compatible stress variation* is of the form

$$\delta\sigma_{ij} = C_{ijkl} \varepsilon_{kl}(\delta u)$$

where δu is the difference of two admissible displacement fields, so that (2.40) is replaced by

$$\delta u_i = 0 \text{ on } S_1 \tag{2.41}$$

Compatible stress variations form a linear vector space C_o. It may be proved that, equipped with the scalar product (2.39), C_o is complete and thus, closed in E. Note that if we know *one* displacement field u_o such that $u_{oi} = \bar{u}_i$ on S_1, any compatible displacement is of the form

$$u = u_o + \Delta u \tag{2.42}$$

where Δu is an admissible displacement variation ($\Delta u = 0$ on S_1).

Let us derive the orthogonal complement S_o of C_o. It is defined by the condition

$$\int_V \sigma_{ij} \varepsilon_{ij} \, dV = 0$$

for each $\varepsilon_{ij} \in C_o$. The condition is

$$\int_V \sigma_{ij} \frac{1}{2}(D_i \delta u_j + D_j u_i)\, dV = 0$$

and expresses that σ_{ij} is a self-stress. S_o, the set of self-stresses is the orthogonal complement S_o of C_o. As C_o is closed, the orthogonal complement of S_o is also C_o.

Now, let us consider a stress field verifying (2.36), i.e.

$$\int_V C^{-1}_{ijkl} \sigma_{kl} \delta\sigma_{ij} dV - \int_{S_1} n_j \delta\sigma_{ji} \overline{u}_i dS = 0 \tag{2.43}$$

for each self-stress $\delta\sigma_{ij}$. Let u_o be a displacement field such that $u_{oi} = \overline{u}_i$ *on* S_1. One has

$$\begin{aligned}\int_V \varepsilon_{ij}(u_o)\delta\sigma_{ij} dV &= \int_V \frac{1}{2}(D_i u_{oj} + D_j u_{oi})\, \delta\sigma_{ij} dV \\ &= \int_{S_1} n_j \delta\sigma_{ji} u_{oj} dS + \int_{S_2} n_j \delta\sigma_{ji} u_{oj} dS - \int_V D_j \delta\sigma_{ji} u_{oj} dV \\ &= \int_{S_1} n_j \delta\sigma_{ji} \overline{u}_j dS\end{aligned}$$

because the two other therms vanish for a self-stress. Condition (2.43) is thus equivalent to

$$\int_V \left[C^{-1}_{ijkl}\sigma_{kl} - \varepsilon_{ij}(u_o)\right] \delta\sigma_{ij} dV = 0$$

or

$$\sigma_{kl} - C_{klij}\varepsilon_{ij}(u_o) \in C_o$$

But this is to say

$$\sigma_{kl} = C_{klij}\varepsilon_{ij}(u_o) + C_{klij}\varepsilon_{ij}(\Delta\, u)$$

or

$$\sigma_{kl} = C_{klij}\varepsilon_{ij}(u_o + \Delta\, u)$$

which is precisely compatibility.

2.9. Particular cases

The preceeding considerations concerned the general case of elasticity. Similar variational principles may be developed for specialized theories, such as beams, plates, bars, by two ways

(i) Set in Hellinger-Reissner the corresponding hypotheses. By this way, it is possible to develop particularized theories. We will follow this way in the case of plates.

(ii) Directly use known expressions of energy. To illustrate this, consider the case of a Navier beam which is clamped at x = 0 and submitted to a lineic load p(x) (fig. 12). The curvature is

$$\chi = \frac{d^2v}{dx^2} = v''$$

if v is the displacement field. The bending moment M is related to the curvature by the equation

$$M = EI\chi$$

where E is Young's modulus and I is the inertia modulus. The lineic energy density is thus given by

$$\delta W = M\delta\chi = EI\chi\delta\chi, \qquad W = EI\frac{\chi^2}{2}$$

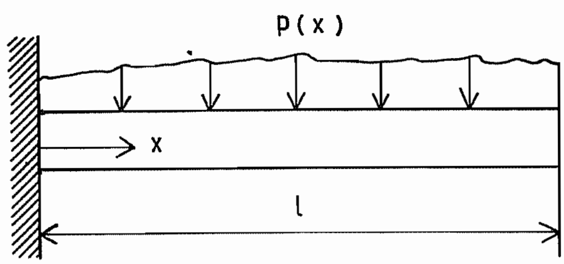


Fig. 12

So, the strain energy is

$$\mathscr{U} = \frac{1}{2}\int_o^l EIv''^2\,dx$$

The potential energy of the load is given by

$$\mathscr{P} = -\int_o^l pv dx$$

The equilibrium of the load is then obtained when the total energy

$$\mathscr{E} = \mathscr{U} + \mathscr{P}$$

is minimum.

Exercise

Show that $\delta\mathscr{E} = 0$ leads to the well known equilibrium equation

$$\frac{d^2 M}{dx^2} = p$$

Suggestion : A double integration by parts has to be performed. Variations at the end of the beam give statical boundary conditions.

CHAPTER 3

FINITE ELEMENTS OF BARS

3.1. Introduction

In this chapter, bar trusses will be envisaged. This very simple case will be used to illustrate the different steps of a matrix structural analysis, avoiding complications related to a more complex problem.

3.2. Bar element

Consider (fig. 13) a bar element of section A, Young's modulus E, and length l. If the displacements at the ends of the bar are q_1 and q_2, the strain energy of the bar is given by

$$\mathcal{U} = \frac{1}{2}\frac{EA}{l}(q_1 - q_2)^2 \tag{3.1}$$

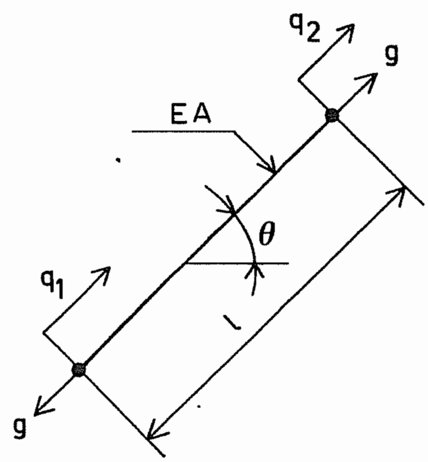


Fig. 13

This may be written

$$\mathcal{U} = [q_1 q_2]\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3.2}$$

with

$$k = \frac{EA}{l} \tag{3.3}$$

The matrix

$$K_{e\,loc} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{3.4}$$

is called the *element stiffness matrix in local axes*. The three last words refer to the fact that the displacements are expressed in the axis of the bar. Note that this matrix is *singular*. In fact,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

This is not troublesome but perfectly logical, because a displacement of the form $q_{loc}^T = [1,1]$ corresponds to a *rigid body motion* which does not produce strain energy.

3.3. Bar stiffness in global axes

Assembling the different bars of a truss supposes the use of an unique axes system at each node. The simplest one is given by displacements u and v along axes x and y. The transformation is (fig. 14)

$$q = u \cos\theta + v \sin\theta \tag{3.5}$$

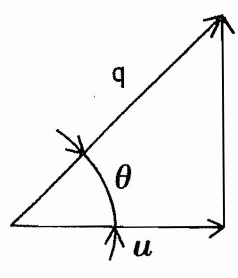


Fig. 14

at each node, and or the bar number e,

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{q_{e,\,loc}} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}}_{T_e} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}}_{q_{e,\,s}} \tag{3.6}$$

where index e refers to element e, and S means structural. The element strain energy is thus given by

$$U = \frac{1}{2} q_{e,S}^T T_e^T K_{eloc} T_e q_{eS} = \frac{1}{2} q_{eS}^T K_{eS} q_{eS} \tag{3.7}$$

This defines the element matrix in structural axes

$$K_{eS} = T_e^T \; K_{e\,loc} \; T_e = k \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \tag{3.8}$$

This matrix is of the same rank as $K_{e\,loc}$, it is 1. In other words, it admits three independant singularities, which are :

$u_1 = 1$, $v_1 = 0$, $u_2 = 1$, $v_1 = 0$: translation along x
$u_1 = 0$, $v_1 = 1$, $u_2 = 0$, $v_1 = 1$: translation along y
$u_1 = 0$, $v_1 = 0$, $u_2 = \sin\theta$, $v_2 = -\cos\theta$: rotation round point 1.

These are the rigid body motions of the bar.

3.4. Assembling the different bars of a truss

To assemble the different elements of the truss illustrated in fig. 15, let us first define the global displacement vector

$$q^T = \left[\underbrace{u_1 v_1}_{node\,1} \; \underbrace{u_2 v_2}_{node\,2} \; u_3 v_3 \; u_4 v_4 \; u_5 v_5 \right] \tag{3.9}$$

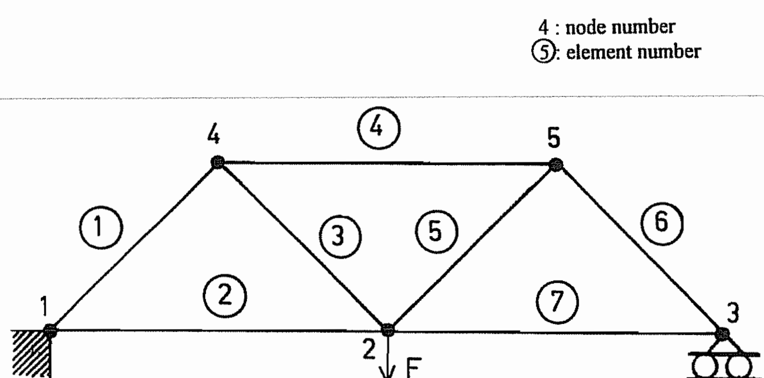


Fig. 15

Each element has 4 displacements which may be indexed in the global displacement vector by an *element localization vector* l_e, defined by the relation

$$q_{ei} = q\,(l_e(i)) \tag{3.10}$$

a fact that is illustrated by the following table

Element	node 1	node 2	$l_e(1)$	$l_e(2)$	$l_e(3)$	$l_e(4)$
1	1	4	1	2	7	8
2	1	2	1	2	3	4
3	2	4	3	4	7	8
4	4	5	7	8	9	10
5	2	5	3	4	9	10
6	3	5	5	6	9	10
7	2	3	3	4	5	6

(To find the numbers of d.o.f., note that at node i, $u_i = \# 2 * i - 1$ and $v_i = \# 2 * i$)

The strain energy is then

$$\mathcal{U} = \frac{1}{2}\sum_e q_{ei} K_{eij} q_{ej} = \frac{1}{2}\sum_e q\,(l_e(i))\; K_{eij}\, q\,(l_e(j)) \tag{3.11}$$

so that the structural matrix is assembled by the following algorithm (N = number of d.o.f.)

```
for i = 1,N and j = 1,N
        K (i,j) ← 0
endfor
For each element e
        For i : 1,4 and j = 1,4
                K(le(i), le(j)) ← K(le(i), le(j)) + Ke(i,j)
        endfor
endfor
```

The following drawing represents the contributions of each element in the matrix

	1	2	3	4	5	6	7	8	9	10
1	①②	①②	②	②			①	①		
2	①②	①②	②	②			①	①		
3	②	②	②③⑤⑦	②③⑤⑦	⑦	⑦	③	③	⑤	⑤
4	②	②	②③⑤⑦	②③⑤⑦	⑦	⑦	③	③	⑤	⑤
5			⑦	⑦	⑥⑦	⑥⑦			⑥	⑥
6			⑦	⑦	⑥⑦	⑥⑦			⑥	⑥
7	①	①	③	③			①③④	①③④	④	④
8	①	①	③	③			①③④	①③④	④	④
9			⑤	⑤	⑥	⑥			⑤⑥	⑤⑥
10			⑤	⑤	⑥	⑥			⑤⑥	⑤⑥

As can be seen, all diagonal elements of the matrix are non-zero. In contrary, a lot of non-diagonal elements vanish.

3.5. Solution of the elastic problem

Let us assume that, as illustrated in fig. 15, there is a load F in the direction of (-v) at node 2. So, the load vector will be g^T : [0,0,0,-F,0,0,0,0,0,0]
We have to solve the equation

$$Kq = g$$

taking account to the fact that $u_1 = 0$, $v_1 = 0$, $v_3 = 0$, it is $q_1 = 0$, $q_2 = 0$, $q_6 = 0$. This may be taken in consideration by suppressing lines and columns of K corresponding to these degrees of freedom. In other words, the final system will be

$$\begin{bmatrix} K_{33} & K_{34} & K_{35} & K_{37} & K_{38} & K_{39} & K_{3,10} \\ K_{43} & K_{44} & K_{45} & K_{47} & K_{48} & K_{49} & K_{4,10} \\ K_{53} & K_{54} & K_{55} & K_{57} & K_{58} & K_{59} & K_{5,10} \\ K_{73} & K_{74} & K_{75} & K_{77} & K_{78} & K_{79} & K_{7,10} \\ K_{83} & K_{84} & K_{85} & K_{87} & K_{88} & K_{89} & K_{8,10} \\ K_{93} & K_{94} & K_{95} & K_{97} & K_{98} & K_{99} & K_{9,10} \\ K_{10,3} & K_{10,4} & K_{10,5} & K_{10,7} & K_{10,8} & K_{10,9} & K_{10,10} \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \\ q_5 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (3.12)$$

3.6. Return to bar loads

Having solved system (3.12), one knows the displacement vector q. Then, in each element

$$g_e = k(q_{e1}-q_{e2})$$

with, at each node,

$$q_e = u \cos\theta + v \sin\theta$$

from which element loads are easy to compute.

CHAPTER 4

BEAM ELEMENTS

4.1. Introduction

This chapter, devoted to beam elements, will give us the opportunity of present in a simple context general methods of finite elements.

4.2. Navier beam

A Navier beam is characterized by the hypothesis that normal sections remain normal during deformation. In other terms, shear deformations are neglected. Let x be the longitudinal coördinate of the beam, and let v(x) be the transverse displacement at coordinate x. Then, the curvature is given by

$$\chi = \frac{d^2 v}{dx^2} = v'' \tag{4.1}$$

The bending moment μ is related to the curvature by

$$\mu = EI\chi \tag{4.2}$$

E being Young's modulus and I, the inertia. The strain energy is thus

$$\mathcal{U} = \frac{1}{2}\int_o^l EIv''^2 dx \tag{4.3}$$

The potential energy is of the form

$$\mathcal{P} = -\int_o^l pvdx$$

where p is the transverse load by unit length.

At the end of the beam, displacements v and rotations $\varphi = v'$ have to be connected.

4.3. Monomial basis

Owing to the fact that a beam element must been connected at both ends by 4 values v(o), φ(o), v(l), φ(l), the simplest polynomial interpolation between these values will have four parameters. This requires at least a *cubic* interpolation of the form

$$v = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 = \underbrace{\left[1, x, x^2, x^3\right]}_{M(x)} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}_{a} \tag{4.4}$$

where a is the vector of parameters and M(x), the basis matrix, which is here composed of monomials. From (4.4), one obtains

$v' = M'(x)\, a = [0,\ 1,\ 2x,\ 3x^2]\, a$
$v'' = M''(x)\, a = [0,\ 0,\ 2,\ 6x]\, a$

and the strain energy may be calculated as

$$\mathcal{U} = \frac{1}{2}\int_o^l EIv''^2\, dx = \frac{1}{2}\, a^T \left(\int_o^l EI \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6x \end{bmatrix} \left[0\ 0\ 2\ 6x\right] dx \right) a$$

$$= \frac{1}{2}\, a^T \left(\int_o^l EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 12x \\ 0 & 0 & 12x & 36x^2 \end{bmatrix} dx \right) a = \frac{1}{2}\, a^T J\, a \tag{4.5}$$

where J is called *integral matrix*. For constant EI, it is given by

$$J = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4l & 6l^2 \\ 0 & 0 & 6l^2 & 12l^3 \end{bmatrix} \tag{4.6}$$

This matrix is two times singular, the singularities being given by the following vectors of parameters :

$a^T = [1\ 0\ 0\ 0]$ ($v = 1$, rigid translation)
$a^T = [0\ 1\ 0\ 0]$ ($v = x$, rigid rotation)

Let us now turn to the potential of the loads. One has

$$\mathcal{P} = -\int_o^l pv dx = -\int_o^l pM(x)\, a = -b^T a \tag{4.7}$$

where

$$b = \int_o^l M^T(x) p \, dx$$

are the generalized loads conjugated to the parameters a. In the present case,

$$b^T = (\beta_1, \beta_2, \beta_3, \beta_4) \tag{4.7b}$$

with

$$\beta_1 = \int_o^l p dx, \beta_2 = \int_o^l px \, dx, \beta_3 = \int_o^l px^2 dx, \beta_4 = \int_o^l px^3 dx \tag{4.8}$$

4.4. Connection

We have now to use generalized displacements which automatically ensure continuity of the displacement and the slope. Such displacements will be

$$q_1 = v(o), q_2 = \varphi(o), q_3 = v(l), q_4 = \varphi(l) \tag{4.9}$$

and are related to the parameters by

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} M(o) \\ M'(o) \\ M(l) \\ M'(l) \end{bmatrix} a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} a = C\, a \tag{4.10}$$

where C is the *connection matrix* which is here square and invertible.
Inverting this relation, one obtains

$$a = C^{-1} q \tag{4.11}$$

from which it is possible to express the strain energy as

$$\mathcal{U} = \frac{1}{2} a^T J a = \frac{1}{2} q^T C^{-T} J C^{-1} q$$

it is

$$\mathcal{U} = \frac{1}{2} q^T K q \tag{4.12}$$

with the stiffness matrix

$$K = C^{-T} J\, C^{-1} \tag{4.13}$$

which, as J, is two time singular. In the same way, the potential energy may be written as

$$\mathcal{P} = -b^T a = -b^T C^{-1} q$$

or

$$\mathcal{P} = -g^T q \tag{4.14}$$

where

$$g = C^{-T} b \tag{4.15}$$

is the vector of generalized loads conjugated to the displacements q.

4.5. Stresses

Suppose now that the problem is assembled and solved, so that displacements are known. It is theoretically possible to obtain local values of the bending moment from

$$\mu = EI\chi = EI\ M''(x)\, a = EI\ M''(x)\, C^{-1} q \tag{4.16}$$

but these values are ordinarily poor approximations because the convergence of derivatives is not so fast as that of the displacements. However, equilibrium is verified for the generalized loads of the form (4.15) and it is therefore logical to use stress values which are related to these loads. Let us consider the loads conjugated to the parameters a,

$$b = C^T g = C^T K q = C^T K C a = J a \tag{4.17}$$

One has

$$\begin{aligned} \delta\mathcal{U} &= \delta a^T J a = \delta a^T \int_o^l M''(x)^T EIM''(x)\, a dx \\ &= \delta\, a^T \int_o^l M''(x)^T \mu\ dx \end{aligned} \tag{4.18}$$

so that

$$b_1 = \int_o^l 0\,\mu\ dx,\ b_2 = \int_o^l 0\,\mu\ dx,\ b_3 = \int_o^l 2\mu\ dx,\ b_4 = \int_o^l 6x\mu\ dx \tag{4.19}$$

A special confidence may thus be attributed to the two means b_3 and b_4. The last one is generally not used and it is a common practice to use the mean value

$$\overline{\mu} = \frac{b_3}{2l} \tag{4.20}$$

To obtain this value from the results, one has to store at the element generation level the so-called *stress matrix* such that

$$\bar{\mu} = Tq \tag{4.21}$$

As

$$\bar{\mu} = \frac{1}{2l}\left[\textit{third line of } J\right] a = \frac{1}{2l}\left[\textit{third line of } J\right] C^{-1} q,$$

one has

$$T = \frac{1}{2l}\left[\textit{third line of } J\right] C^{-1} \tag{4.22}$$

4.6. Shape functions

Shape functions are an alternative to the preceeding developments using the monomial basis. The idea is to directly write

$$v(x) = N_1(x)q_1 + N_2(x)q_2 + N_3(x)q_3 + N_4(x)q_4 = N(x)q \tag{4.23}$$

where q_1, q_2, q_3, q_4 have the same meaning as before (see (4.9)). This is theoretically equivalent, as

$$v(x) = M(x)\ a = M(x)\ C^{-1}\ q \tag{4.24}$$

so that

$$N(x) = M(x)\ C^{-1} \tag{4.25}$$

However, the most popular way consists to directly write the *shape functions* $N_i(x)$. To perform this, note that if $q_1 = q_2 = 0$, one has

$$v(x) = x^2\left[A + B\,(l - x)\right] \tag{4.26}$$

from which

$$v(l) = l^2 A, \quad A = \frac{v(l)}{l^2} \tag{4.27}$$

Now,

$$v'(x) = 2x\ A + (2lx - 3x^2)B$$

and

$$\varphi(l) = 2lA - l^2 B$$
$$B = 2\frac{v(l)}{l^3} - \frac{\varphi(l)}{l^2} \tag{4.28}$$

So, if $q_1 = q_2 = 0$,

$$\begin{aligned} v(x) &= v(l)\frac{x^2}{l^2} + 2\frac{v(l)}{l^3}(lx^2 - x^3) - \frac{\varphi(l)}{l^2}(lx^2 - x^3) \\ &= v(l)\left[3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right] + \varphi(l)\left[\frac{x^3}{l^2} - \frac{x^2}{l}\right] \end{aligned} \tag{4.29}$$

Similarly, if $q_3 = q_4 = 0$,

$$v(x) = (l - x)^2 \left[A + Bx\right] \tag{4.30}$$

from which

$$v(o) = l^2 A, \quad A = \frac{v(o)}{l^2} \tag{4.31}$$

Since

$$v'(x) = 2\,(x - l)\,\left[A + Bx\right] + (l - x)^2 B,$$

one has

$$\varphi(o) = -2lA + l^2 B, \quad B = \frac{\varphi(o)}{l^2} + 2\frac{v(o)}{l^3} \tag{4.32}$$

and, if $q_3 = q_3 = 0$;

$$v(x) = v(o)\,\frac{(l - x)^2}{l^2} + 2v(o)\,\frac{x(l - x)^2}{l^3} + \frac{\varphi(o)}{l^2}\,x(l - x)^2$$

$$= v(o)\left[1 - 4\frac{x}{l} + 5\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right] + \varphi(o)\left(x - 2\frac{x^2}{l} + \frac{x^2}{l^2}\right) \tag{4.33}$$

Summing results (4.29) and (4.33), one obtains

$$\begin{cases} N_1(x) = 1 - 3\dfrac{x^2}{l^2} + 2\dfrac{x^3}{l^3} \\ N_2(x) = x - 2\dfrac{x^2}{l} + \dfrac{x^3}{l^2} \\ N_3(x) = 3\dfrac{x^2}{l^2} - 2\dfrac{x^3}{l^3} \\ N_4(x) = \dfrac{x^3}{l^2} - \dfrac{x^2}{l} \end{cases} \tag{4.34}$$

To construct the stiffness matrix, we have now to write

$$v'' = N''_1(x)q_1 + N''_2(x)q_2 + N''_3(x)q_3 + N''_4 q_4 = N''q \tag{4.35}$$

so that

$$\mathcal{U} = \frac{1}{2}q^T \int_o^l EI N''^T N'' dx q = \frac{1}{2}q^T K q \tag{4.36}$$

with

$$K = \int_o^l EI\, N''^T\, N'' dx \tag{4.37}$$

This is of course more direct than the preceeding formulation, but N" is not so easy to compute as M" so that errors are more likely.

Exercise

Compute K explicitely

4.7. Computing generalized loads from shape functions

The potential energy is now

$$\mathcal{P} = -\int_o^l pv dx = -\int_o^l pN(x) dx q$$

so that

$$g = \int_o^l N^T(x)\, p\, dx \tag{4.38}$$

or explicitely,

$$g_1 = \int_o^l p(x) \cdot \left(1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right) dx$$

$$g_2 = \int_o^l p(x) \cdot \left(x - 2\frac{x^2}{l} + \frac{x^3}{l^2}\right) dx$$

$$g_3 = \int_o^l p(x) \cdot \left(3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right) dx$$

$$g_4 = \int_o^l p(x) \cdot \left(\frac{x^3}{l^2} + \frac{x^2}{l}\right) dx$$

It is interesting to compute these four values in the case p = ct. Then,

$$g_1 = pl\left(1 - \frac{3}{3} + \frac{2}{4}\right) = \frac{pl}{2}$$

$$g_2 = pl^2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{pl^2}{12}(6 - 8 + 3) = \frac{pl^2}{12}$$

$$g_3 = pl\left(\frac{3}{3} - \frac{2}{4}\right) = \frac{pl}{2}$$

$$g_4 = pl^2\left(\frac{1}{4} - \frac{1}{3}\right) = -\frac{pl^2}{12}$$

At this stage, it is highly recommended to verify that these loads give the correct value of work for rigid body notions. This permits to avoid most errors.

a) Translation $q_1 = 1$, $q_2 = 0$, $q_3 = 1$, $q_4 = 0$

$$\mathcal{T} = pl$$

$$\mathcal{T} = g_1 + g_2 = \frac{pl}{2} + \frac{pl}{2}$$

b) Rotation about the mid-point $x = l/2, q_1 = -l/2, q_2 = 1, q_3 = l/2, q_4 = l$

$$\mathcal{T} = o$$

$$\mathcal{T} = -\frac{l}{2}g_1 + g_2 + \frac{l}{2}g_3 + g_4 = -\frac{pl^2}{4} + \frac{pl^2}{12} + \frac{pl^2}{4} - \frac{pl^2}{12} = 0$$

4.8. Comparison monomial basis versus shape functions

Shape functions are more popular than the monomial basis. However, the monomial basis is by far more flexible and permits the development of some elements which in the shape function frame would be unthinkable. Moreover, the procedures with the monomial basis may be automatized (see appendix) avoiding the high risks of algebraic errors which are common with shape functions. For this reason, the author largely prefers the use of the monomial basis, even if it seems not so direct a method as the shape functions.

CHAPTER 5

FINITE ELEMENTS FOR PLANE ELASTICITY

5.1. The two plane states in elasticity

Recall that there are two plane states in elasticity, which are *plane strain* and *plane stress.*

a) *Plane strain* is characterized by the assumptions

$$\varepsilon_{13} \equiv 0,\ \varepsilon_{23} \equiv 0,\ u_3 \equiv 0 \tag{5.1}$$

Taking account of the last one, the two others give

$$D_3u_1 = -D_1u_3 = 0$$
$$D_3u_2 = -D_2u_3 = 0$$

So, hypotheses (5.1) are equivalent to

$$u_1 = u_1(x_1, x_2),\ u_2 = u_2(x_1, x_2),\ u_3 = 0 \tag{5.2}$$

Now, in the isotropic case, Hooke's law is given by

$$\sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{ll}\delta_{ij}\right] \tag{5.3}$$

where G is Coulomb's modulus. From this follows

$$\sigma_{11} = \frac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_{11} + \nu\varepsilon_{22}\right]$$

$$\sigma_{22} = \frac{2G}{1-2\nu}\left[(1-\nu)\varepsilon_{22} + \nu\varepsilon_{11}\right]$$

$$\sigma_{33} = \frac{2G}{1-2\nu}\left[\nu(\varepsilon_{11} + \varepsilon_{22})\right] = \nu(\sigma_{11} + \sigma_{22})$$

$$\tau_{12} = 2G\,\varepsilon_{12} = G\gamma_{12} \text{ with } \gamma_{12} = 2\varepsilon_{12}$$

$$\tau_{13} = \tau_{23} = 0$$

The strain energy density is then

$$W = \frac{1}{2}(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \tau_{12}\gamma_{12}) = e^T He \tag{5.4}$$

with the *strain vector*

$$e^T = (\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}) \tag{5.5}$$

and the *Hooke matrix*

$$\text{H (plane strain)} = \frac{2G}{1-2\nu}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{5.6}$$

The derivative of the energy density by respect of the strain vector is the *stress vector*

$$\mathrm{s} = \mathrm{H\ e} \tag{5.7}$$

with

$$\mathrm{s}^{\mathrm{T}} = (\sigma_{11}, \sigma_{22}, \sigma_{12}) \tag{5.8}$$

Relation (5.7) represents Hooke's law for plane strain.

b) *Plane stress* is characterized by the assumptions

$$\sigma_{13} = 0,\ \sigma_{23} = 0,\ \sigma_{33} = 0 \tag{5.9}$$

Using inverse Hooke's law

$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{ll}\delta_{ij}\right],$$

one obtains

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\sigma_{22}\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{11}\right]$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \text{ or } \gamma_{12} = \frac{2(1+\nu)}{E}\sigma_{12}$$

$$\varepsilon_{13} = \varepsilon_{23} = 0$$

$$\varepsilon_{33} = \frac{1}{E}\left[-\nu(\sigma_{11} + \sigma_{22})\right]$$

This may be inverted as

$$\sigma_{11} = \frac{E}{1-\nu^2}\,(\varepsilon_{11} + \nu\varepsilon_{22})$$

$$\sigma_{22} = \frac{E}{1-\nu^2}\,(\varepsilon_{22} + \nu\varepsilon_{11})$$

$$\sigma_{12} = \frac{E}{2(1+\nu)}\,\gamma_{12} = \frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\cdot\gamma_{12}$$

that is (5.7) with the following Hooke's matrix :

$$\text{H (plane stress)} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{5.10}$$

At this stage, it is interesting to compare these two matrices. A mean stiffness index may be the trace of each matrix whis, as is well known, is the sum of the eigenvalues of the matrix. In the case of plane train, one has

$$\text{trace H (plane strain)} = \frac{2G}{1-2\nu}\left[(1-\nu)+(1-\nu)+\left(\frac{1}{2}-\nu\right)\right] = \frac{2G\left(\frac{5}{2}-3\nu\right)}{(1-2\nu)}$$

and, taking account of the fact that

$$G = \frac{E}{2(1+\nu)}$$

this leads to

$$\text{trace H (plane strain)} = \frac{E\left(\frac{5}{2}-3\nu\right)}{(1+\nu)\,(1-2\nu)} \tag{5.11}$$

In the case of plane stress, one obtains

$$\text{trace H (plane stress)} = \frac{E}{1-\nu^2}\left[1+1+\frac{1}{2}-\frac{\nu}{2}\right] = \frac{E}{1-\nu^2}\left[\frac{5}{2}-\frac{\nu}{2}\right] \tag{5.11b}$$

The ratio between these two values is

$$\lambda = \frac{trace\ H(plane\ strain)}{trace\ H(plane\ stress)} = \frac{\left(\frac{5}{2} - 3\nu\right)}{\left(\frac{5}{2} - \frac{\nu}{2}\right)} * \frac{1-\nu}{1-2\nu} \tag{5.12}$$

In the classical case $\nu = 0.3$, this gives

$$\lambda = \frac{2.5-0.9}{2.5-0.15} * \frac{0.7}{0.4} = 1{,}191$$

This means that plane strain is on average 19 % more siff than plane stress for $\nu = 0.3$. The ratio λ strongly depends on Poisson's ratio, as indicated in the following table

ν	0	0.1	0.2	0.3	0.4	0.45	0.49	0.499	0.4999	0.5
λ	1	1.010	1.056	1.191	1.696	2.780	11.65	111.6	1112	∞

When $\nu \to 0.5$, both behaviors completely diverge. This is due to the fact that in plane strain, incompressibility imposes

$$\varepsilon_{11} + \varepsilon_{22} = 0.$$

Treating incompressible materials in plane strain or in 3-dimensional elasticity requires specialized formulations that go beyond the scope or the present course. For the interested reader, see [8, 9, 10, 11, 12].

5.2. Triangular elements

Triangular elements will be first considered, as they are the simplest ones. The monomial basis method will be followed because it is the most general one. Shape functions for these problems will be developed later.

5.2.1.- P_k-development

We will call P_k the space of complete polynomials of degree k, which are as follows :

$$\underbrace{\alpha_1}_{\text{degree } 0} + \underbrace{\alpha_2 x + \alpha_3 y}_{\text{degree } 1} + \underbrace{\alpha_4 x^2 + \alpha_5 xy + \alpha_6 xy^2}_{\text{degree } 2} + \underbrace{\alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3}_{\text{degree } 3} +$$

$$+ \underbrace{\alpha_{11} x^4 + \alpha_{12} x^3 y + \alpha_{13} x^2 y^2 + \alpha_{14} xy^3 + \alpha_{15} y^4}_{\text{degree } 4} + \ldots \text{ etc } \ldots$$

The terms of each strict degree are arranged in descending order for x and corresponding ascending order or y. As can be seen, a complete polynomial of degree k depends to coefficients whose number is

$$n_a = \sum_{i=o}^{k}(i+1) = \frac{[1+(k+1)].(k+1)}{2} = \frac{(k+1)\,(k+2)}{2} \qquad (5.13)$$

5.2.2.- Displacement expressions

In each element, an expression of the displacements will be chosen, which is a complete polynomial of degree k. So, at degree 2,

$$\begin{aligned} u_1 &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \\ u_2 &= \alpha_7 + \alpha_8 x + \alpha_9 y + \alpha_{10} x^2 + \alpha_{11} xy + \alpha_{12} y^2 \end{aligned} \qquad (5.14)$$

or, equivalently,

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = M(x,y)\, a \qquad (5.15)$$

with, in the case (5.14)

$$a^T = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}] \qquad (5.16)$$

and

$$M(x,y) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \qquad (5.17)$$

It is clear from (5.13) that for a degree k, displacements depend on n_a parameters, with

$$n_a = 2 * \frac{(k+1)\,(k+2)}{2} \qquad (5.18)$$

5.2.3.- Strains

Strains are related to displacements by

$$\begin{aligned} \varepsilon_{11} &= D_1 u_1 \\ \varepsilon_{22} &= D_2 u_2 \\ \gamma_{12} &= D_1 u_2 + D_2 u_1 \end{aligned}$$

and this may be systematized in the form

$$e=\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix}=\begin{bmatrix}D_1 & 0\\ 0 & D_2\\ D_2 & D_1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\partial u \tag{5.19}$$

where ∂ is the differential operator

$$\partial=\begin{bmatrix}D_1 & 0\\ 0 & D_2\\ D_2 & D_1\end{bmatrix} \tag{5.20}$$

Applying this operator to the displacement expression (5.15) leads to

$$\mathrm{e}=\partial\,\mathrm{M}\,\mathrm{a}=\mathrm{B}\,\mathrm{a} \tag{5.21}$$

where $\mathrm{B}=\partial\,\mathrm{M}$ will be explicitely, for a second-degree displacement field

$$B=\begin{bmatrix}D_1 & 0\\ 0 & D_2\\ D_2 & D_1\end{bmatrix}\begin{bmatrix}1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2\end{bmatrix}$$

or

$$B=\begin{bmatrix}0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y\\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0\end{bmatrix} \tag{5.22}$$

5.2.4.- Strain energy

The strain energy is now given by

$$\mathcal{U}=\frac{1}{2}\int_V e^T H e dV,$$

where H is Hooke's matrix. Using expression (5.2) of the strains, one obtains

$$\mathcal{U}=\frac{1}{2}a^T J a \tag{5.23}$$

where

$$J = \int_V B^T H \, B \, dV \tag{5.24}$$

is called the integral matrix of the element. This matrix is three time singular, due to the fact that the strain is zero for the three rigid body displacements

$u_1 = 1,$	$u_2 = 0$	(rigid translation along x)
$u_1 = 0$	$u_2 = 1$	(rigid translation along y)
$u_1 = y$	$u_2 = -x$	(rigid rotation round the origin)

5.2.5.- Connection. What has to be connected ?

All preceeding developments concerned one isolated element. Now, the structure is componed by a lot of adjacent elements which have to be connected in some manner. Here lies the main originality of the finite element method, which is to *choose the weakest connections that ensure a correct definition of the strain energy*. To obtain the proper criterium, let us note that the strains are obtained from the displacements by applying the differentiation operator ∂ which is in the general case of order m (here, m = 1, but for plates, it will be 2). Strains appear as squared in the strain energy, and the H matrix is constituted of bounded terms. So, existency of the strain energy requires *that the displacements and their derivatives up to the order m are square-integrable*, a fact that may be written

$$\|u\|^2_{m,V} = \int_V \left[u^2 + \sum_i (D_i u)^2 + \sum_{kl} (D_{kl} u)^2 + \dots \right] dV < \infty \tag{5.25}$$

$\rightarrow$ up to order m

The space of functions that possess this property is known *as SOBOLEV space of order m and is generally noted $H^m(\Omega)$* [13,14]. When is this property verified ? A correct proof has been given by CIARLET [15], but here, an intuitive reasoning will be used. In each element, the displacements are of the C^∞ class, because they are polynomials. Let us adopt a progression line which is transverse to the interface I of two adjacent elements e_1 and e_2 (fig. 16). If a C^o-continuity at this interface is assumed, the derivative along the progression line will be discontinuous but nevertheless square-integrable. But the second derivative exhibits a Dirac measure on the interface, which is not square-integrable (fig. 17).

From this, we induce the following rule, *to ensure that $u \in H^m(V)$, it is necessary to impose $u \in C^{m-1}$*. This is the *conformity condition.*

Elements that are of the C^P-class are said

- *strictly conforming* if p = m-1
- *non conforming* if p < (m-1)
- *more than conforming* if p ≥ (m-1) and derivatives of greater order than (m-1) are connected.

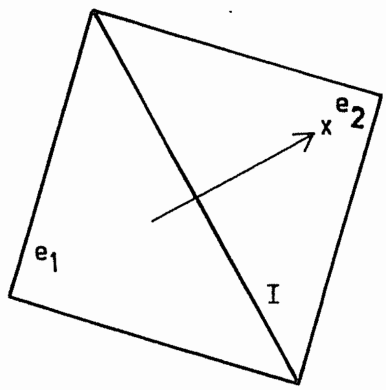


Fig. 16

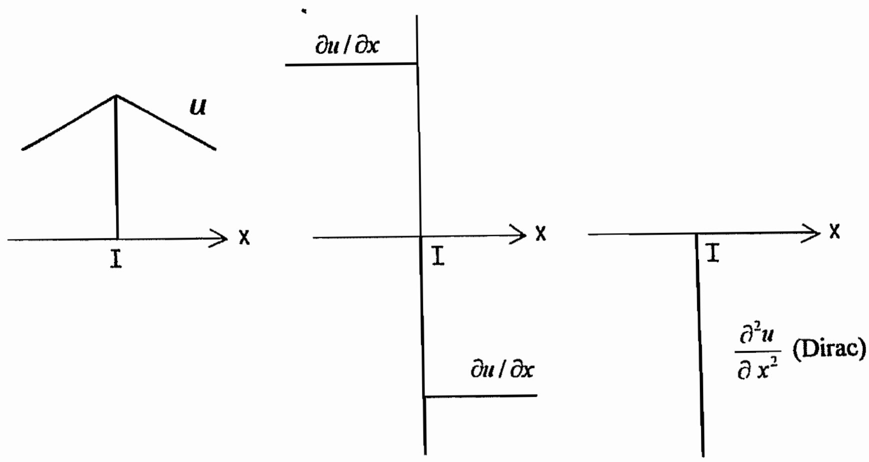


Fig. 17

Nonconforming elements are violating variational rules and, except some special cases, do not converge if the mesh is refined. More than conforming elements are admissible, but in problems whose solution is not very regular, their convergence is slowed down by the excess of conformity. This may be illustratd by the simple case of a solution of the roof-type (fig. 18) which is exactly represented by two C^{o}-elements and necessits, with C^{1} elements, a lot of elements to be correctly approximated.

The best strategy is thus to ensure a strict conformity, in each case where it is possible.

5.2.6.- Defining connectors

The general method to ensure conformity consists to define some degrees of freedom on the interface whose connection guarantees the correct connection. These degrees of freedom are called *generalized displacements* or *connectors.* In the present case, field u_1 and u_2 are independant and may thus be treated separately.

The connection strategy for triangles is a consequence of the property of complete polynomials of degree k to be such polynomials in any cartesian system, orthogonal or not. The trace of a complete polynomial of degree k on any straight line is thus also a polynomial of degree k of the current coordinate of this line. So (fig. 19),

(i) First degree – Connection at the three nodes ensures C^{o}-continuity
(ii) Second degree – On each interface, a 2d-degree field necessitates 3 values to be uniquely defined. We have thus to add to the nodal values one other one on each interface. The classical choice is the mid-point value.
(iii) Third degree – Add to the nodal values 2 other ones on the interfaces, logically at 1/3 and 2/3 of the interface length.
(iv) Fourth degree – Three supplementary values on each interface.

And so on. Thus, the number of required connectors for degree k is

$$n_{q_1 field} = 3 + 3\,(k-1) = 3k$$

for each field. As we have here two fields u_1 and u_2, the number of connectors will be

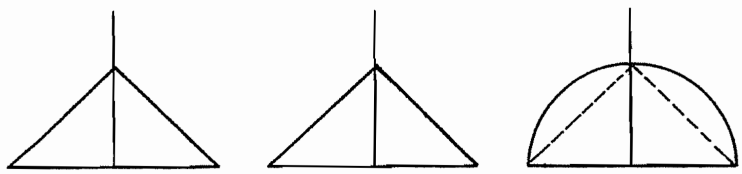
$$n_q = 2{*}3k \tag{5.26}$$

5.2.7.- The connection matrix

Adopting the order of connectors

$q^T = [u_1\ v_1\ u_2\ v_2\ \ldots\ u_{3k}\ v_{3k}]$.

The relation between the connector and the parameters may be written from (5.14)



Roof function $\in H^1$

Interpolation by two C°-elements

Interpolation by two C^1-elements

Fig. 18

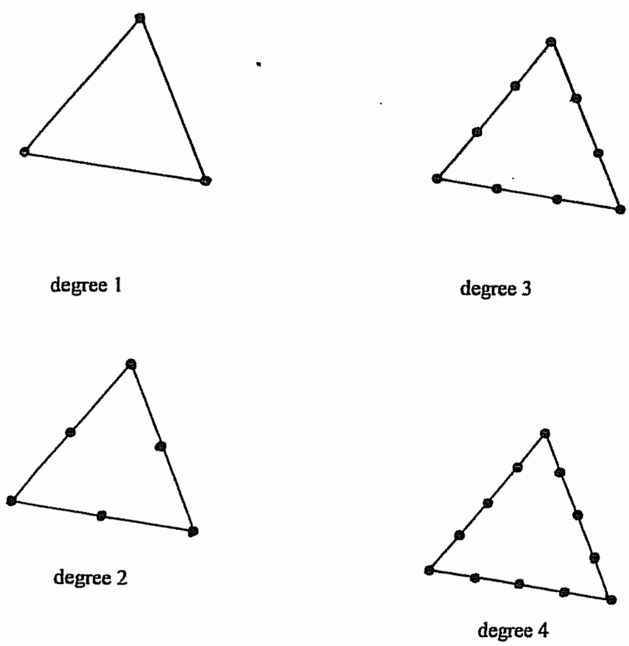


Fig. 19 : Connections

$$q = \begin{bmatrix} u_1 \\ \underline{v_1} \\ u_2 \\ \underline{u_2} \\ \vdots \\ u_{3k} \\ v_{3k} \end{bmatrix} = \begin{bmatrix} M(x_1,y_1) \\ \hline M(x_2,y_2) \\ \hline \vdots \\ \hline M(x_{3k},y_{3k}) \end{bmatrix} a = C\,a \tag{5.27}$$

where C is the *connection matrix*. Let us first suppose that this matrix is square and invertible. Then, it will be possible to write

$$a = C^{-1} q \tag{5.28}$$

and the strain energy may be expressed as

$$\mathcal{U} = \frac{1}{2} a^T J a = \frac{1}{2} q^T C^{-T} J C^{-1} q$$

or

$$\mathcal{U} = \frac{1}{2} q^T K q \tag{5.29}$$

where the stiffness matrix is given by

$$K = C^{-T}\, J\, C^{-1} \tag{5.30}$$

Unfortunately, the connection matrix is not necessarily square, because the connectors and parameters have been chosen independently. Moreover, even square, it could be singular due to an inadequate choice of connectors. The connection matrix has n_q lines and n_a columns. So, it will be square if and only if $n_q = n_a$. Noting that

$$n_q = 2{*}3k\,, \qquad n_a = 2 * \frac{(k+1)\,(k+2)}{2} \tag{5.31}$$

it is clear that n_a generally differs from n_q, as reported in the following table

K	n_a	n_q	n_a-n_q
1	2*3	2*3	0
2	2*6	2*6	0
3	2*10	2*9	2*1
4	2*15	2*12	2*3
5	2*21	2*15	2*6

Generally speaking, $(n_a - n_q)$ is given by

$$n_a - n_q = 2*\left[\frac{(k+1)(k+2)}{2} - 3k\right]_+ = 2*\left[\frac{k^2+3k+3-6k}{2}\right]_+ = 2*\left[\frac{(k-1)(k-2)}{2}\right]_+ \quad (5.32)$$

the index + meaning "positive part" (i.e. if $x \leq 0$, $x_+ = 0$).

So, from degree 3, the connection matrix is of the horizontal rectangular type, that is, it has less lines than columns. There are thus *at least* $(n_a - n_q)$ solutions of the equation

$$Ca = 0 \quad (5.33)$$

Such solutions are called *bubble modes* because they represent polynomials which vanish on the element boundary and have therefore the aspect of a bubble. Let n_b be their number. If $n_b = n_a - n_q$, the matrix C is of maximum rank n_q. But it is conceivable that n_b would be greater than $n_a - n_q$. In this case, the rank of the connection matrix would not be n_q, but $n_q - n_s$, with

$$\text{rank} = n_q - n_s = n_a - n_b,$$

it is to say

$$n_s = n_q - n_a + n_b \quad (5.34)$$

A strictly positive value of n_s would mean that there exists some dependency between the connectors which can be expressed by k independent solutions of

$$C^T l = 0 \quad (5.35)$$

In this case, it is known from algebra that the system

$$C\,a = q$$

cannot have a solution unless the vector q verifies

$$l^T q = 0$$

for each solution of (5.35). Other displacement vectors, which are combinations of the n_s independant vectors l, cannot be related to the parameters and may be interpreted as *spurious kinematical modes.*

The question is now to find the exact number of bubble modes. Equation (5.33) means that each field, which is a polynomial of degree k, vanishes on the element boundary. This boundary consists in three straight lines whose equations are

$$c_1(x,y) = 0,\ c_2(x,y) = 0,\ c_3(x,y) = 0 \quad (5.36)$$

c_1, c_2 and c_3 being first-order polynomials. Any polynomial P_k vanishing on the boundary is necessarily of the form

$$P_k = c_1.c_2.c_3.P_{k-3} \tag{5.37}$$

where P_{k-3} is an arbitrary polynomial of degree (k-3). The number of bubble modes for each field is thus equal to the number of parameters of a polynomial of degree (k-3), that is

$$\left[\frac{(k-2)\,(k-1)}{2}\right]_+$$

For two fields, the number of bubble modes is thus

$$n_b = 2*\left[\frac{(k-1)\,(k-2)}{2}\right]_+ \tag{5.38}$$

which is precisely equal to $(n_a - n_q)$. The conclusion is that *the connection matrix is here of the maximum rank* n_q.

5.2.8.- How to treat bubble modes

Bubble modes are thus equal to zero on the boundary of the element. The classical way to treat them is to define supplementary points inside the element and to add to the displacement vector the displacement q_b at these points. The extended connection equation is then

$$q_* = \begin{bmatrix} q \\ q_b \end{bmatrix} = \begin{bmatrix} C \\ C_b \end{bmatrix} a = C_*\, a \tag{5.39}$$

The number of elements of q_b has naturally to be equal to n_b. The interior points have to he choosen adequately, so that the new square matrix C_* is invertible. The condition for this is that

$$\begin{Bmatrix} Ca = 0 \\ C_b a = 0 \end{Bmatrix} \Rightarrow a = 0 \tag{5.40}$$

or equivalently, the double condition

$$\begin{Bmatrix} u \text{ is a bubble} \\ q_b = 0 \end{Bmatrix} \Rightarrow u = 0 \tag{5.41}$$

In the present case, a proper choice of points defining the bubbles is given by the intersections of parallels to each side from the connection points (fig. 20).

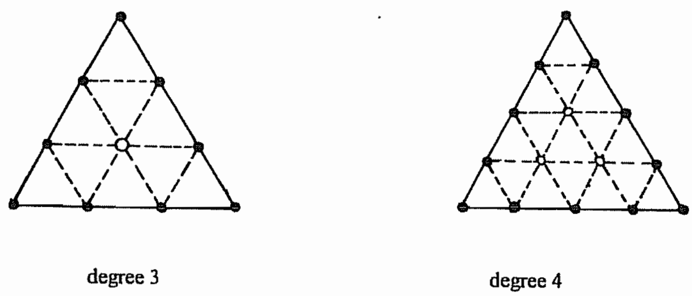


Fig. 20 : Interior points defining the bubbles

Relation (5.39) may then be inverted,

$$a = C_*^{-1} q_* \tag{5.42}$$

from which the strain energy becomes

$$\mathcal{U} = \frac{1}{2} q_*^T K_* q_* \tag{5.43}$$

with

$$K_* = C_*^{-T} J C_*^{-1} \tag{5.44}$$

5.2.9.- Surface loads

We have now to compute the potential energy of loads that are eventually reported along the surface of the element. This energy is given by

$$\mathcal{P} = -\int_S (f_1 u_1 + f_2 u_2)\, dS = -\int_S f^T u dS \tag{5.45}$$

Using the expression (5.15) of the displacement field, one immediately obtains

$$\mathscr{P} = -\left(\int_S f^T M dS\right) a = -b^T a \tag{5.46}$$

with

$$b = \int_S M^T f dS \tag{5.47}$$

These are the loads conjugated to the parameters. Now, the connection relation

$a = C^{-1}q,$

or, when bubbles are present,

$a_* = C_*^{-1}q_*,$

leads to

$$\mathscr{P} = -b^T C^{-1} q = -g^T q \text{ or } \mathscr{P} = -b^T C_*^{-1} q_* = -g_*^T q_* \tag{5.48}$$

respectively, with

$$g = C^{-T}b \quad \text{or} \quad g_* = C_*^{-T}b \tag{5.49}$$

5.2.10.- Side loads

At the boundary, there generally exists side loads corresponding to a potential energy of the form

$$\mathscr{P} = -\int_\Gamma (t_1 u_1 + t_2 u_2)\, ds = -\int_\Gamma t^T u ds \tag{5.50}$$

In most cases, there is only one side of each element which is concerned and it is common practice to assume that it is the first side of the element (i.e. the side which corresponds to the two first nodes of the element definition).

On this side, a local lineic coordinate s may be used. If k is the degree of the displacement, its trace on this side will be a polynomial of the same degree in terms of s. So, on the side,

$$\begin{cases} u_1 = \alpha_1 + \alpha_2 s + \alpha_3 s^2 + \ldots + \alpha_{k+1} s^k \\ u_2 = \alpha_{k+2} + \alpha_{k+3} s + \ldots + \alpha_{2k+2} s^k \end{cases}$$

or, in condensed form,

$$u = M_{side}(s) \,.\, a_{side} \tag{5.51}$$

Let q_{side} be the subset of connectors lying on the concerned side. One has thus a connection of the form

$$q_{side} = C_{side}\, a_{side} \tag{5.52}$$

which is always square and invertible. Then,

$$\mathscr{P} = -\int_S t^T M_{side} ds . a_{side} = -b_{side}^T a_{side}$$

with

$$b_{side} = \int_S M_{side}^T t \, ds \tag{5.53}$$

and, after inversion of (5.52),

$$\mathscr{P} = -b_{side}^T C_{side}^{-1} q_{side} = -g_{side}^T q_{side} \tag{5.54}$$

with

$$g_{side} = C_{side}^{-T} b_{side} \tag{5.55}$$

The fact that q_{side} is a subset of q may be written

$$q_{side} = L\, q \tag{5.56}$$

where L is a proper rectangular matrix. Then,

$$\mathscr{P} = -g_{side}^T L q = -g^T q$$

with

$$g = L^T g_{side} \tag{5.57}$$

This generalized force vector has to be added to an eventual surface load vector.

5.2.11.- Stresses

When the elastic problem is solved for the displacements, it is necessary to return to each element in order to compute the stresses. This work is prepared at the generation level, where *element stress matrices* are computed and stored on a peripherical memory. After resolution of the displacements, it will suffice to use these matrices to compute the stresses.

Stress calculation is a very delicate problem because local values of the stresses do not converge as fast as displacements so that in most problems, they are of poor quality, and exhibit strong discontinuities at the interelement boundaries. This problem is the subject of many researches at the present time, including a *posteriori* error measures, but such developments go far beyond the scope of the present course.

As already said about beams, generalized loads g are in equilibrium, so that any stress result coming from these loads may be considered with a good confidence. We know that

$$g = Kq = C^{-T} Ja \text{ or } g_* = K_* q_* = C_*^{-T} Ja \tag{5.58}$$

following the fact that there are bubble modes or not. So,

$$b = C^T g \text{ or } b = C_*^T g \tag{5.59}$$

respectively verifies

$$b = Ja \tag{5.60}$$

Now, the variation of the element strain energy is given by

$$\delta \mathcal{U} = \int_S s^{-T} \partial \delta u dS = \int_S s^T B dS \delta a = b^T \delta a$$

so that

$$b = \int_S B^T s \, dS \tag{5.61}$$

represents some mean values of the stresses. As an example, at the second degree, one obtains from (5.22)

$$b_1 = 0$$
$$b_2 = \int \sigma_{11} \, dS$$
$$b_3 = \int \sigma_{12} \, dS$$
$$b_4 = 2\int x \, \sigma_{11} \, dS$$
$$b_5 = \int (y\sigma_{11} + x\sigma_{12}) \, dS$$
$$b_6 = 2\int y \, \sigma_{12} \, dS$$
$$b_7 = 0$$
$$b_8 = \int \sigma_{12} \, dS$$
$$b_9 = \int \sigma_{22} \, dS$$
$$b_{10} = 2\int x \, \sigma_{22} \, dS$$

$$b_{11} = \int (x\,\sigma_{22} + y\,\sigma_{12})\, dS$$
$$b_{12} = 2\int y\,\sigma_{22}\, dS \qquad (5.62)$$

In the simplest version, only mean values of the stresses are retained, which are

$$\bar{\sigma}_{11} = \frac{1}{A} b_2$$
$$\bar{\sigma}_{22} = \frac{1}{A} b_9$$
$$\bar{\sigma}_{12} = \frac{1}{A} b_8$$

where A is the area of the element. Since b = Ja

$$\bar{s} = \begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{bmatrix} = Sa \qquad (5.63)$$

where

$$S = \frac{1}{A}\begin{bmatrix} 2d\ line\ of\ J \\ 9th\ line\ of\ J \\ 8th\ line\ of\ J \end{bmatrix} \qquad (5.64)$$

Now, using the connection matrix,

$$\bar{s} = Tq \text{ or } \bar{s} = T_* q_* \qquad (5.65)$$

with

$$T = SC^{-1} \text{ or } T_* = SC_*^{-1} \qquad (5.66)$$

5.2.12.- Condensation of bubble modes

Recall that the elastic problem consists to minimize the total energy $\mathcal{E} = \mathcal{U} + \mathcal{P}$. The total energy of the structure is the sum of contributions of each element. When minimizing $\mathcal{E}$ by respect to a displacement, one obtains an equilibrium equation which relates this displacement to the set of all other ones that are directly connected to it by an element. In the particular case of a bubble, there is no direct interaction with other elements, and the minimization process may be performed at the element level.

The element total energy is of the form

$$\mathscr{E} = \frac{1}{2} q_c^T K_{cc} q_c + q_b^T K_{bc} q_c + \frac{1}{2} q_b^T K_{bb} q_b - g_c^T q_c - g_b^T q_b \tag{5.67}$$

where q_c represents connected displacement, and q_b, bubble modes. Varying these ones, one obtains

$$\frac{\partial \mathscr{E}}{\partial q_b} = K_{bc} q_c + K_{bb} q_b - g_b = 0$$

In this relation, the square matrix K_{bb} is positive definite because when the displacement vanishes on the boundary, the strain energy which has the expression

$$\frac{1}{2} q_b^T K_{bb} q_b$$

has to be positive. One may thus write

$$q_b = K_{bb}^{-1} g_b - K_{bb}^{-1} K_{bc} q_c \tag{5.68}$$

Using this result, one obtains successively

$$\begin{aligned}
q_c^T K_{cb} q_b &= q_c^T K_{cb} K_{bb}^{-1} g_b - q_c^T K_{cb} K_{bb}^{-1} K_{bc} q_c \\
\frac{1}{2} q_b^T K_{bb} q_b &= \frac{1}{2} g_b^T K_{bb}^{-1} g_b - g_b^T K_{bb}^{-1} K_{bc} q_c + \frac{1}{2} q_c^T K_{cb} K_{bb}^{-1} K_{bc} q_c \\
- g_b^T q_b &= - g_b^T K_{bb}^{-1} g_b + g_b^T K_{bb}^{-1} K_{bc} q_c
\end{aligned}$$

so that

$$\begin{aligned}
\min_{q_b} \mathscr{E} &= \frac{1}{2} q_c^T (K_{cc} - K_{cb} K_{bb}^{-1} K_{bc})\, q_c - (g_c - K_{cb} K_{bb}^{-1} g_b)^T q_c - \frac{1}{2} g_b^T K_{bb}^{-1} g_b \\
&= \frac{1}{2} q_c^T \widetilde{K}_{cc} q_c - \widetilde{g}_c^T q_c + \mathscr{E}_o
\end{aligned} \tag{5.69}$$

where

$$\begin{cases}
\widetilde{K}_{cc} = K_{cc} - K_{cb} K_{bb}^{-1} K_{bc} \\
\widetilde{g}_c = g_c - K_{cb} K_{bb}^{-1} g_b \\
\mathscr{E}_o = -\dfrac{1}{2} g_b^T K_{bb}^{-1} g_b
\end{cases} \tag{5.70}$$

The term $\mathscr{E}_0$ has no further influence on the solution of the problem, but has to be taken in account when computing the energy.

Concerning the stress matrix, the relation is

$$s = T_c q_c + T_b q_b = T_c q_c - T_b K_{bb}^{-1} g_b - T_b K_{bb}^{-1} K_{bc} q_c$$

or

$$s = \widetilde{T}_c q_c + s_o \tag{5.71}$$

with

$$\begin{cases} \widetilde{T}_c = T_c - T_b K_{bb}^{-1} K_{bc} \\ s_o = -T_b K_{bb}^{-1} g_b \end{cases} \tag{5.72}$$

The last term s_o represents the contribution of the bubble modes to stresses.

5.2.13.- Convergence

Finite element converge in an energetical sense. Let us define the energetical norm

$$\|u\|^2 = \int_V e^T(u)\, H\, e(u)\, dV \tag{5.73}$$

It may then be proved that, provided the exact solution has derivatives up to the order (k+1) which are square-integrable, then the approximate solution u_h veriries

$$\|u - u_h\| \leq C(u).h^{k+1-m} \tag{5.74}$$

where h is the largest element diameter and m is the order of derivatives entering in the operator ∂ (here, m = 1). If the exact solution is not so regular, say, it admits square integrable derivatives up to a certain order l > m, then

$$\|u - u_h\| \leq C(u)\, h^{l-m} \tag{5.75}$$

In the extreme case where l = m, one may prove that

$$\|u - u_h\| \rightarrow 0 \text{ when h} \rightarrow 0, \tag{5.76}$$

but no order of convergence can be guaranteed. These results suppose that no element has too low on angle, a fact that may be taken into consideration when generating a mesh.

The boundary is in general curvilinear, so that the use of rectilinear elements induces a supplementary error which is O ($h^{3/2}$). Therefore, there is no real reason to use elements of a degree greater than 2.

Note that in the present case, polynomials are used to represent the displacements. But it is conceivable to use other functions. In such a case, the two following requirements are known as *completeness conditions.*

1st requirement : *rigid body motions* have to be present in the element
2nd requirement : *constant strain nodes* have also to be represented exactly

Strictly speaking, these conditions are only necessary in the limiting case $h \to 0$. But experiences show that their exact verification improves accuracy. These conditions were first stated by BAZELEY, CHEUNG, IRONS and ZIENKIEWICZ [16]. In the case of polynomials, they are automatically verified for a degree $k \geq m$.

5.3. Rectangular elements

5.3.1.- Connections

Let us first examine the required connections on a rectangle. For each field, if it is assumed that *the degree is k on each side*, it is necessary to ensure the connection on each node and on (k-1) points on each interface. This gives

$$n_q = 4 + 4\,(k - 1) = 4\,k \tag{5.77}$$

connectors for each field (fig. 21).

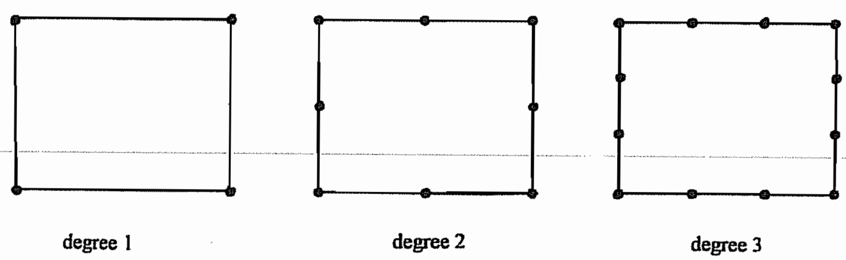


Fig. 21 : Connections of a rectangle

5.3.2.- Displacement field

The displacement field will be choosed to be of degree k on each parallel to axes x and y. This condition is verified by so-called Q_k polynomials, which are products of k^{th} degree polynomials of x and of y :

$$\text{Degree 1} : u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$\text{Degree 2} : u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + y\,(\alpha_4 + \alpha_5 x + \alpha_6 x^2) + y^2\,(\alpha_7 + \alpha_8 x + \alpha_9 x^2)$$
$$\begin{aligned}\text{Degree 3} : u = {} & \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \\ & + y\,(\alpha_5 + \alpha_6 x + \alpha_7 x^2 + \alpha_8 x^3) \\ & + y^2\,(\alpha_9 + \alpha_{10} x + \alpha_{11} x^2 + \alpha_{12} x^3) \\ & + y^3\,(\alpha_{13} + \alpha_{14} x + \alpha_{15} x^2 + \alpha_{16} x^3)\end{aligned}$$

and so on.

The number of parameters of such a field is visibly

$$n_a = (k+1)^2 \tag{5.78}$$

and is in general not equal to n_q. The difference is

$$n_a - n_q = k^2 + 2k + 1 - 4k = (k-1)^2$$

is the minimal number of bubble modes. To find these, note that the sides of the rectangle have equations

$$c_1(x) = (x-a) = 0, \quad c_2(x) = (y-b) = 0, \quad c_3(x) = (x-c) = 0, \quad c_4(x) = (y-d) = 0$$

so that a bubble mode is of the form

$$Q_k = (x-a)\,(x-c)\,(y-c)\,(y-b)\,Q_{k-2}\,(x)$$

and the number of independant bubbles is equal to the number of parameters of a Q_{k-2} polynomial, that is

$$n_b = (k-1)_+^2 \tag{5.79}$$

We may thus conclude that

$$n_s = n_q - n_a + n_b = 0$$

and the only question is now to define proper internal displacements. A correct choice is to use the values of the displacement at intersections of parallels to x and y passing through nodes (fig. 22).

All other considerations about triangles apply here with obvious modifications.

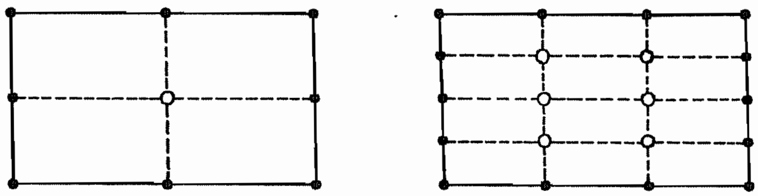


Fig. 22 : Bubbles on a rectangle

5.4. Parallelogram elements

The case of a parallelogram may be treated similarly by using oblique axes x and y for the coordinates (fig. 23). Each field is then of the form

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

at the first degree,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + y\,(\alpha_4 + \alpha_5 x\ \alpha_6 x^2) + y^2\,(\alpha_7 + \alpha_8 x + \alpha_9 x^2)$$

at the second degree, and so on. This case is very similar as the rectangle, except the fact that one has to compute

$$\frac{\partial u}{\partial X} \text{ and } \frac{\partial u}{\partial Y}$$

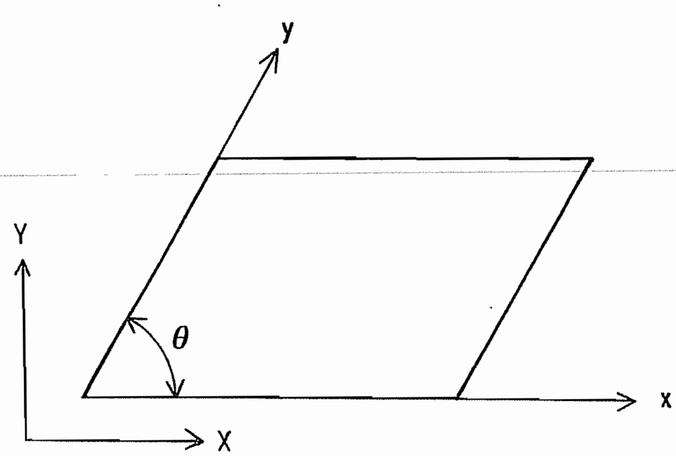


Fig. 23 : Parallelogram

For this, note that

$$X = x + y\cos\theta \qquad\qquad Y = y\sin\theta$$

so that

$$J = \frac{\partial(X,Y)}{\partial(x,y)} = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \tag{5.80}$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}$$

or

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = J^T \begin{bmatrix} \dfrac{\partial u}{\partial X} \\ \dfrac{\partial u}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial X} \\ \dfrac{\partial u}{\partial Y} \end{bmatrix} \tag{5.81}$$

from which

$$\begin{bmatrix} \dfrac{\partial u}{\partial X} \\ \dfrac{\partial u}{\partial Y} \end{bmatrix} = J^{-T} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\cot\theta & \dfrac{1}{\sin\theta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} \tag{5.82}$$

From this follows

$$\varepsilon_{11} = \frac{\partial u_1}{\partial X} = \frac{\partial u_1}{\partial x}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial y} = -\cot\theta\frac{\partial u_2}{\partial x} + \frac{1}{\sin\theta}\frac{\partial u_2}{\partial y}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial Y} + \frac{\partial u_2}{\partial X} = -\cot\theta\frac{\partial u_1}{\partial x} + \frac{1}{\sin\theta}\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}$$

it is, the operator ∂ is given by

$$\partial = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & -\cot\theta\dfrac{\partial}{\partial x} + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial y} \\ -\cot\theta\dfrac{\partial}{\partial x} + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \tag{5.83}$$

Here, a technical problem arises, because the matrix B = ∂M is no more composed of monomials, but of polynomials. Now, the procedure that automatically computes the matrix J from pseudo-formal expressions of polynomials is based on the fact that B should be composed of mononials (see appendix). To circumvene this technical difficulty, the following trick may be used. Let us note that Hooke's law

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \gamma_{12} \\ \varepsilon_{22} \end{bmatrix}$$

may be written equivalently

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{22} & H_{23} \\ H_{21} & H_{22} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} \\ \dfrac{\partial u_1}{\partial X_2} \\ \dfrac{\partial u_2}{\partial X_1} \\ \dfrac{\partial u_2}{\partial X_2} \end{bmatrix} = H^* \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} \\ \dfrac{\partial u_1}{\partial X_2} \\ \dfrac{\partial u_2}{\partial X_1} \\ \dfrac{\partial u_2}{\partial X_2} \end{bmatrix} \tag{5.84}$$

or

$$s^* = H^* e^* \tag{5.85}$$

where

$$s^{*T} = (\sigma_{11}, \sigma_{12}, \sigma_{12}, \sigma_{22}) \tag{5.86}$$

and

$$e^{*T} = \left(\frac{\partial u_1}{\partial X_1}, \frac{\partial u_1}{\partial X_2}, \frac{\partial u_2}{\partial X_1}, \frac{\partial u_2}{\partial X_2} \right) \tag{5.87}$$

Now, the strain energy density is given by

$$\delta W = \frac{1}{2} s^{*T} e^{*} = \frac{1}{2} e^{*T} H^{*} e^{*}$$

Furthermore,

$$e^{*} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} \\ \frac{\partial u_1}{\partial X_2} \\ \frac{\partial u_2}{\partial X_1} \\ \frac{\partial u_2}{\partial X_2} \end{bmatrix} = \begin{bmatrix} J^{-T} & 0 \\ 0 & J^{-T} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \end{bmatrix} \tag{5.88}$$

so that

$$\delta W = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix} H^{**} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \end{bmatrix} \tag{5.89}$$

with the constant matrix

$$H^{**} = \begin{bmatrix} J^{-1} & 0 \\ 0 & J^{-1} \end{bmatrix} H \begin{bmatrix} J^{-T} & 0 \\ 0 & J^{-T} \end{bmatrix} \tag{5.90}$$

A similar procedure may also be used in the case of isoparametric elements, see chapt. 6.

5.5. General quadrilateral elements

The preceeding concepts do not give the possibility of developing a general quadrilateral element. The only way is thus to cut the element in two triangles. However, general quadrilaterals may be obtained from an isoparametric transformation (see chapter 6).

5.6. Shape functions

Up to now, the monomial basis was systematically used. There is however another way to develop elements, by making use of *shape functions*. Our discussion will be made with only one field, but this is by no means restrictive.

The fundamental idea is as follows. The combination of

$$u = M\,(x,y)\,a$$

and

$$a = C^{-1}q$$

(with eventually bubbles contained in q) leads to

$$u = M\,C^{-1}q = Nq$$

with

$$N = MC^{-1} \tag{5.91}$$

Explicitely, for one field,

$$u = N_1q_1 + N_2q_2 + N_3q_3 + \ldots \tag{5.91b}$$

These N_i's are called *shape functions*. At point number j, one has

$$u\,(x_j,y_j) = q_j = N_1\,(x_j,y_j)\,q_1 + N_2\,(x_j,y_j)\,q_2 + \ldots$$

from which follows

$$N_i\,(x_j,y_j) = \delta_{ij} \tag{5.92}$$

This relation allows to find the shape functions by an inspection method. Let us illustrate this in some cases.

5.6.1.- Third degree shape functions on a triangle

Let us first consider the first degree polynomial $c_1\,(x,y)$ which is defined by the conditions

$$c_1\,(x_1,y_1) = 1,\;\; c_1\,(x_2,y_2) = 0,\;\; c_1\,(x_3,y_3) = 0 \tag{5.93}$$

This polynomial being of the form

$c_1(x,y) = Ax + By + C$

the three conditions are

$$\begin{cases} Ax_1 + By_1 + C = 1 \\ Ax_2 + By_2 + C = 0 \\ Ax_3 + By_3 + C = 0 \end{cases}$$

The determinant of this system is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

which is equal to ± 2* area of the triangle. This system thus always admits a solution.

Function c_1 is equal to zero along the side 2-3 (fig. 25). On line 5-8, it is equal to 1/3, and on the line 4-9, its value is 2/3.

One may define similarly c_2 and c_3. These functions are called *area coordinates* of the triangle. In fact, for any point P(x,y) of the triangle, one has (fig. 24)

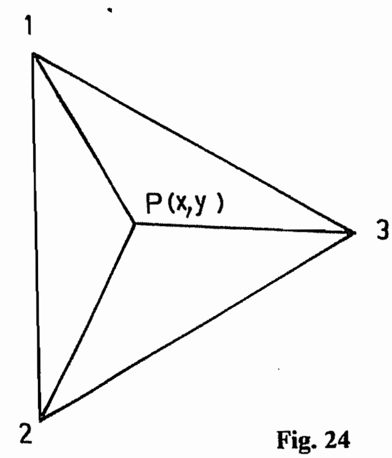


Fig. 24

$$c_1(x,y) = \frac{area\ P-2-3}{area\ 1-2-3}$$

$$c_2(x,y) = \frac{area\ P-1-3}{area\ 1-2-3}$$

$$c_3(x,y) = \frac{area\ P-1-2}{area\ 1-2-3}$$

so that

$$c_1(x,y) + c_2(x,y) + c_3(x,y) = 1 \qquad (5.94)$$

These three functions make the research of shape functions easy. Let us consider (fig. 25) the third degree triangle.

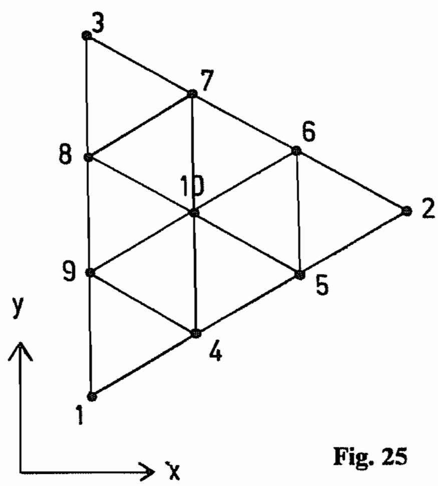


Fig. 25

Node 1 :

$N_1 = 0$ on 2-3 of equation $c_1(x) = 0$
on 5-8 of equation $c_1(x) = 1/3$
on 4-9 of equation $c_1(x) = 2/3$

We may thus write

$$N_1 = \alpha\, c_1 . \left(c_1 - \frac{1}{3}\right) \cdot \left(c_1 - \frac{2}{3}\right),$$

α having to be fixed so that

$$1 = N_1(x_1, y_1) = \alpha \, . \, 1 \, . \, \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}\alpha$$

Finally,

$$N_1 = \frac{9}{2} c_1 \left(c_1 - \frac{1}{3}\right)\left(c_1 - \frac{2}{3}\right) \qquad (5.95)$$

Node 4 :

N_4 is zero at points 1,5,2,9,8,3,6,7. It vanishes thus

on lines 1-3 of equation $c_2 = 0$
2-3 of equation $c_1 = 0$
5-8 of equation $c_1 = 1/3$

So,

$$N_4 = \alpha c_1 \left(c_1 - \frac{1}{3} \right) c_2 ,$$

α being determined by the condition that $N_4 = 1$ at point 4, where

$c_1 = 2/3$ and $c_2 = 1/3$. This leads to

$$1 = \alpha . \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{27} \alpha$$

Therefore,

$$N_4 = \frac{27}{2} c_1 \left(c_1 - \frac{1}{3} \right) c_2 \tag{5.96}$$

The procedure is the same for the other shape functions.

Exercise

Determine the other shape functions and draw them.

5.6.2.- Second degree shape functions on a rectangle

Let us first recall the classical theory of Lagrange interpolation. If f is a function whose values are known at some points $x_1, ..., x_n$, the unique interpolation by a $(n - 1)^{th}$ degree is given by

$$\tilde{f}(x) = \sum_{i=1}^{n} L_i(x)\, f(x_i) \tag{5.97}$$

with

$$L_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} \tag{5.98}$$

The n functions $L_i(x)$, which are knowed as *Lagrange interpolation polynomials*, verify the conditions

$$L_i(x_j) = \delta_{ij}. \tag{5.99}$$

Now, any rectangle of sides a and b may be transformed in the unit square by setting

$$x = a\xi,\ y = b\eta \tag{5.100}$$

We are thus concerned with shape functions on the unit square. Let us consider the important case of a second degree element. On the segment [0,1] (fig. 26), the three Lagrange polynomials corresponding to the points 0, $\frac{1}{2}$, 1 are

$$L_1(\zeta) = \frac{(\zeta - 1)\left(\zeta - \frac{1}{2}\right)}{(0-1)\left(0 - \frac{1}{2}\right)},\quad L_2(\zeta) = \frac{(\zeta - 0)\,(\zeta - 1)}{\left(\frac{1}{2} - 0\right)\left(\frac{1}{2} - 1\right)},\quad L_3(\zeta) = \frac{(\zeta - 0)\left(\zeta - \frac{1}{2}\right)}{(1-0)\left(1 - \frac{1}{2}\right)} \tag{5.101}$$

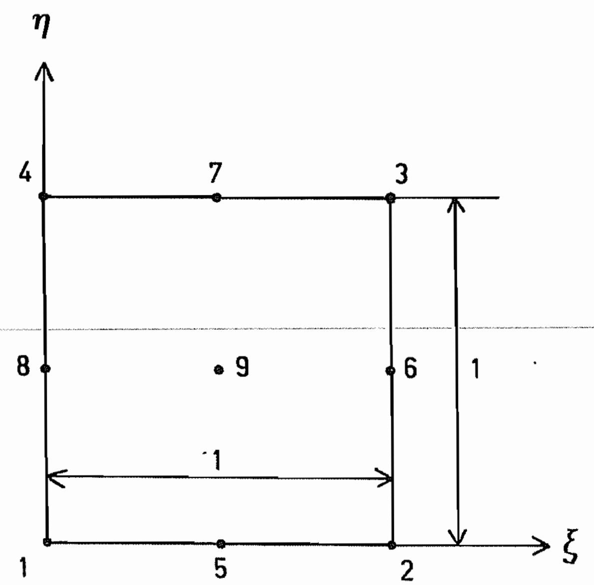


Fig. 26

The shape functions are then the following products

Point n^r i	Coordinates	Shape function N_i
1	(0,0)	$L_1(\xi)L_1(\eta)$
2	(1,0)	$L_3(\xi)L_1(\eta)$
3	(1,1)	$L_3(\xi)L_3(\eta)$
4	(0,1)	$L_1(\xi)L_3(\eta)$
5	$\left(\frac{1}{2},0\right)$	$L_2(\xi)L_1(\eta)$
6	$\left(1,\frac{1}{2}\right)$	$L_3(\xi)L_2(\eta)$
7	$\left(\frac{1}{2},1\right)$	$L_2(\xi)L_3(\eta)$
8	$\left(0,\frac{1}{2}\right)$	$L_1(\xi)L_2(\eta)$
9	$\left(\frac{1}{2},\frac{1}{2}\right)$	$L_2(\xi)L_2(\eta)$

Exercise

Determine the shape functions of a 3d-degree rectangle with bubbles.

5.6.3.- Second degree shape functions on a rectangle, without bubble

It is also possible, by a more subtle choice of shape functions, to avoid the existence of a bubble. The technique of determining such shape functions consists to (fig. 27)

a) Isolate the point O.

b) The shape function is zero on each side which does not contain the concerned point O.

c) Eventually, some points remain, where the shape function has to vanish. Join these by a straight line.

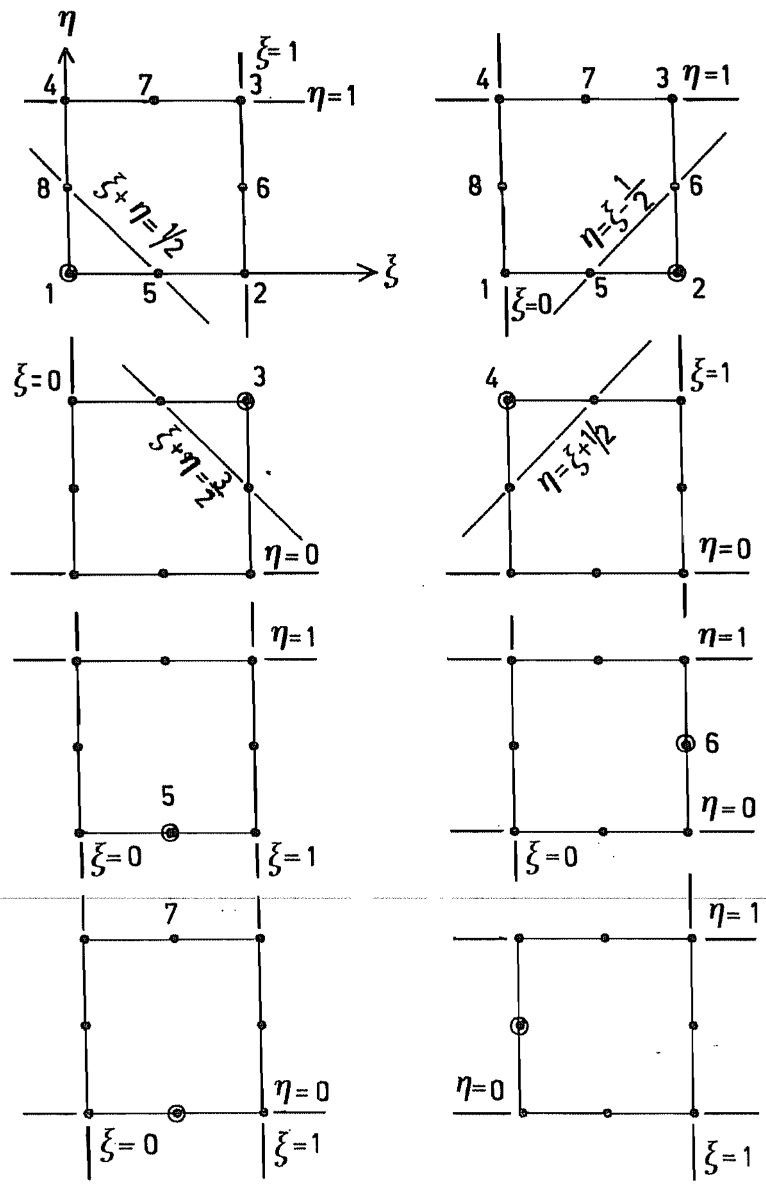


Fig. 27

d) The shape function is then the product of the equations of the three lines defined in b) and c), affected by a coefficient chosen to have the value 1 at the considered point.

Explicitely

- $$N_1 = \alpha\,(1-\xi)\,(1-\eta)\left(\frac{1}{2}-\xi-\eta\right)$$
$$N_1(0,0) = \frac{\alpha}{2} = 1, \qquad \alpha = 2$$
$$N_1 = 2(1-\xi)\,(1-\eta)\left(\frac{1}{2}-\xi-\eta\right)$$

- $$N_2 = \alpha\,\xi\,(1-\eta)\left(\xi-\eta-\frac{1}{2}\right)$$
$$N_2(1,0) = \frac{\alpha}{2} = 1, \qquad \alpha = 2$$
$$N_2 = 2\xi\,(1-\eta)\left(\xi-\eta-\frac{1}{2}\right)$$

- $$N_3 = \alpha\xi\eta\left(\xi+\eta-\frac{3}{2}\right)$$
$$N_3(1,1) = \frac{\alpha}{2} = 1, \qquad \alpha = 2$$
$$N_3 = 2\xi\eta\left(\xi+\eta-\frac{3}{2}\right)$$

- $$N_4 = \alpha(1-\xi)\,\eta\left(\eta-\xi-\frac{1}{2}\right)$$
$$N_4(0,1) = \frac{\alpha}{2} = 1, \qquad \alpha = 2$$
$$N_4 = 2(1-\xi)\,\eta\left(\eta-\xi-\frac{1}{2}\right)$$

- $$N_5 = \alpha\,\xi\,(1-\zeta)\,(1-\eta)$$
$$N_5\left(\frac{1}{2},0\right) = \frac{\alpha}{4} = 1, \qquad \alpha = 4$$
$$N_5 = 4\xi\,(1-\xi)\,(1-\eta)$$

- $$N_6 = \alpha\xi\eta\,(1-\eta)$$
$$N_6\left(1,\frac{1}{2}\right) = \frac{\alpha}{4} = 1, \qquad \alpha = 4$$
$$N_6 = 4\xi\eta\,(1-\eta)$$

- $N_7 = \alpha\xi\,(1-\xi)\,\eta$

$$N_7\left(\frac{1}{2},1\right) = \frac{\alpha}{4} = 1, \qquad \alpha = 4$$

$$N_7 = 4\xi\,(1-\xi)\,\eta$$

- $N_8 = \alpha(1-\xi)\,\eta\,(1-\eta)$

$$N_8\left(0,\frac{1}{2}\right) = \frac{\alpha}{4} = 1, \qquad \alpha = 4$$

$$N_8 = 4(1-\xi)\,\eta\,(1-\eta)$$

Exercise

Find the 12 shape functions of a 3d-degree square without bubbles
Hint : There exists a circle passing through all interface nodes.

5.7. Using shape functions to determine the nodal contributions of a surface load

We saw in section 5.2.9. how to treat a surface load in the context of a monomial basis. Shape functions give an alternative way. In fact, for a load f acting on a displacement u, the potential energy is of the form

$$\mathcal{P} = -\int_S f u dS = -\int_S f\left(\sum_i N_i q_i\right) dS = -\sum_i g_i q_i\,,$$

from which

$$g_i = \int_S N_i f\,dS \tag{5.102}$$

This expression permits an easy computation of the nodal contribution of simple loads. Let us consider the second degree rectangle with 9 nodes, submitted to a constant surface load. From (5.101)

$$\begin{aligned} L_1(\zeta) &= 2\zeta^2 - 3\zeta + 1 \\ L_2(\zeta) &= 4\,(\zeta - \zeta^2) \\ L_3(\zeta) &= 2\zeta^2 - \zeta \end{aligned} \tag{5.103}$$

and

$$I_1 = \int_o^1 L_1(\zeta)\, d\zeta = \frac{1}{6}$$

$$I_2 = \int_o^1 L_2(\zeta)\, d\zeta = \frac{2}{3} \qquad (5.104)$$

$$I_3 = \int_o^1 L_3(\zeta)\, d\zeta = \frac{1}{6}$$

Now,

$$\int_S N_i f dS = f \int_0^1 \int_0^1 N_i ab\, d\xi d\eta,$$

and, using the table of shape functions of section 5.6.2, one obtains

$$\begin{aligned}
g_1 &= f\ ab\ I_1^2 = f\ ab/36 \\
g_2 &= f\ ab\ I_1 I_3 = f\ ab/36 \\
g_3 &= f\ ab\ I_3^2 = f\ ab/36 \\
g_4 &= f\ ab\ I_1 I_3 = f\ ab/36 \\
g_5 &= f\ ab\ I_1 I_2 = f\ ab/9 \\
g_6 &= f\ ab\ I_2 I_3 = f\ ab/9 \\
g_7 &= f\ ab\ I_2 I_3 = f\ ab/9 \\
g_8 &= f\ ab\ I_1 I_2 = f\ ab/9 \\
g_9 &= f\ ab\ I_2^2 = 4f\ ab/9
\end{aligned} \qquad (5.105)$$

Let us insist on the fact that any such computation has to be checked, because errors are likely. The first check will be that the resultant load has the correct value,

$$g_1 + g_2 + \ldots + g_9 = f\,ab.$$

Some symmetry may also be expected in the repartition.

Exercises

1. Determine the 8 nodal contributions of a constant surface load on a 8-node second degree rectangle (see section 5.6.3.).

   There is an artifex. Starting from the preceeding result, one has

   $$w_9 = \frac{1}{2}(w_5 + w_6 + w_7 + w_8) - \frac{1}{4}(w_1 + w_2 + w_3 + w_4) \qquad \text{(see § 5.6.3.)}$$

$$g_9 w_9 = \frac{4fab}{9} w_9 = \frac{2fab}{9}(w_5 + w_6 + w_7 + w_8) - \underbrace{\frac{fab}{9}}_{\frac{4fab}{36}}(w_1 + w_2 + w_3 + w_4)$$

$$g_1^{\bullet} = g_2^{\bullet} = g_3^{\bullet} = g_4^{\bullet} = \frac{fab}{36} - \frac{4fab}{36} = -\frac{3fab}{36} = -\frac{fab}{12}$$

$$g_5^{\bullet} = g_6^{\bullet} = g_7^{\bullet} = g_8^{\bullet} = \frac{fab}{9} + \frac{2fab}{9} = \frac{3fab}{9}$$

Total : $\frac{4\times 3fab}{9} - \frac{4fab}{12} = \frac{4}{3}fab - \frac{1}{3}fab = fab$

2. Do the same with triangles, using the formula

$$\int_\Delta C_1^m C_2^n C_3^p dS = 2S \cdot \frac{m!n!p!}{(m+n+p+2)!}$$

(To prove this formula, note that

$$\int_\Delta C_1^m C_2^n C_3^p dS = 2S\int_\Delta (1-x-y)^m x^n y^p dx\, dy =$$

$$2S\int_o^1 x^n dx \underbrace{\int_o^{1-x}((1-x)-y)^m y^P dy}_{(1-x)^{m+p+1}\int_o^1 (1-z)^m z^p dy} =$$

$$\left(pose\ z = \frac{y}{1-x}\right)$$

$$2S\ B(m+1, m+p+2)B(m+1, p+1) =$$

$$2S.\frac{\Gamma(m+1)\Gamma(m+p+2)}{\Gamma(m+n+p+3)}\frac{\Gamma(m+1)\Gamma(p+1)}{\Gamma(m+p+2)})$$

5.8. Interface shape functions and nodal contributions of an interface load

The same procedure may be used to determine the nodal contributions of an interface load. The problem is here simpler because one-dimensional. We will treat any examples.

5.8.1.- Constant load on second degree interface using the mid-point value of the displacement (fig. 28).

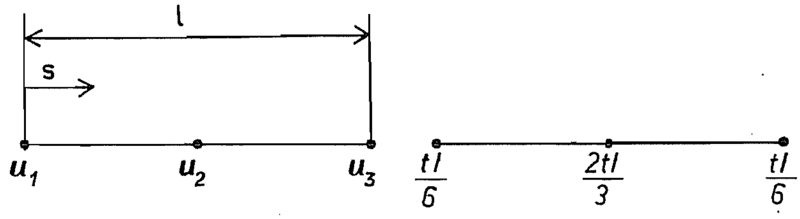


Fig. 28

The interface being of length l, the three degrees of freedom will be $u_1 = u(o)$, $u_2 = u(l/2)$, $u_3 = u(l)$. The shape functions are given by (5.103), with $\zeta = s/l$, so that if t is the reported load,

$$g_1 = tl\int_o^1 L_1(\zeta)\, d\,\zeta = tlI_1 = tl/6$$

$$g_2 = tl\int_o^1 L_2(\zeta)\, d\,\zeta = tlI_2 = 2tl/3$$

$$g_3 = tl\int_o^1 L_3(\zeta)\, d\,\zeta = tlI_3 = tl/6 \qquad (5.106)$$

The correct value of the resultant is easily checked.

Exercise

Same thing, 3[d] and 4[th] degree.

5.8.2.- Same problem, but replacing u_2 by the mean value of the displacement (fig. 29)

Such a degree of freedom will be encountered in the following. So,

$$u_1 = u(o), \quad \tilde{u}_2 = \frac{1}{l}\int_o^l u(s)ds, \quad u_3 = u(l).$$

The three shape functions are determined as follows ($\zeta = s/l$).

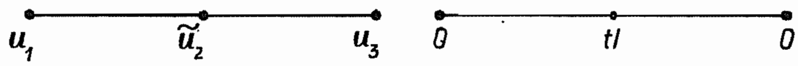


Fig. 29

a) $N_1(\zeta)$ verifies $N_1(1) = 0$ and $\int_o^1 N_1 \, dS = 0$. From the first condition,

$$N_1 = (1 - \zeta)(\alpha + \beta\zeta)$$

and from the second,

$$0 = \alpha \int_o^1 (1-\zeta)\, d\zeta + \beta \int_o^1 (\zeta - \zeta^2)\, d\zeta = \frac{\alpha}{2} + \frac{\beta}{6}, \qquad \beta = -3\alpha.$$

So,

$$N_1 = \alpha\,(1 - \zeta)(1 - 3\zeta) = \alpha\,(1 - 4\zeta + 3\zeta^2)$$

and from the condition $N_1(0) = 1$, $\alpha = 1$. Finally,

$$N_1 = 1 - 4\zeta + 3\zeta^2 \tag{5.107}$$

b) $N_2(\zeta)$ vanishes at $\zeta = 0$ and $\zeta = 1$, so that

$$N_2 = \alpha\,(\zeta - \zeta^2)$$

The value of α is found from the condition

$$1 = \int_o^1 N_2 d\zeta = \alpha \left(\frac{1}{2} - \frac{1}{3} \right) = \alpha / 6 .$$

Thus,

$$N_2 = 6\,(\zeta - \zeta^2) \tag{5.108}$$

c) N_3 may be obtained by replacing ζ by $(1 - \zeta)$ in N_1. This leads

$$N_3 = 1 - 4\,(1 - \zeta) + 3\,(1 - 2\zeta + \zeta^2) = -\,2\zeta + 3\zeta^2 \tag{5.109}$$

For a constant side load t, the nodal contributions will now be

$$g_1 = tl \int_o^1 N_1 d\zeta = 0$$

$$g_2 = tl \int_o^1 N_2 d\zeta = tl$$

$$g_3 = tl \int_o^1 N_3 d\zeta = 0, \tag{5.110}$$

a very pretty result.

5.9. Strain energy from shape functions

It is also possible to obtain the strain energy from the development

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & \dots \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & \dots \end{bmatrix} q \tag{5.111}$$

Applying operator ∂ leads to

$$e = \begin{bmatrix} D_1N_1 & 0 & D_1N_2 & 0 & D_1N_3 & 0 \\ 0 & D_2N_1 & 0 & D_2N_2 & 0 & D_2N_3 \\ D_2N_1 & D_1N_2 & D_2N_1 & D_1N_2 & D_2N_3 & D_1N_3 \end{bmatrix} q = B_N q \tag{5.112}$$

an expression which seems more direct than the method exposed in sections (5.2.3.) and (5.2.4.). However, each term of B_N is a polynomial whose coefficients have to be computed *analytically*. Errors are here very likely and when existing, very difficult to detect in a written program. In contrary, the monomial basis permits a complete automation of the computations, avoiding so any risk of error. Moreover, there exist some complex elements in which the shape function method is not feasible.

CHAPTER 6

ISOPARAMETRIC ELEMENTS

6.1. Introduction

Classical elements, as exposed before, suffer from severe limitations

a) There is no possibility to obtain a general quadrilateral element
b) The boundary, when curved, is badly represented by the polygonal approximation which is unavoidable with classical elements. For this reason, second degree elements do not converge as h^2 but as $h^{3/2}$.

The solution of these problems may be found in coordinate transformations. Up to now, linear or affine transformations were used implicitely. In fact,

- any triangle may be obtained from the unit rectangular triangle by the transformation (fig. 30)

$$\begin{cases} x = x_1 + \xi\,(x_2 - x_1) + \eta\,(x_3 - x_1) \\ y = y_1 + \xi\,(y_2 - y_1) + \eta\,(y_3 - y_1) \end{cases} \qquad (6.1)$$

  a fact that is often used in generation routines

- any rectangle may be obtained from the unit square by the transformation (fig. 31)

$$\begin{cases} x = x_1 + \xi\,(x_2 - x_1) + \eta\,(x_4 - x_1) \\ y = y_1 + \xi\,(y_2 - y_1) + \eta\,(y_4 - y_1) \end{cases} \qquad (6.2)$$

- we already used oblique coordinates to develop a parallelogram element in section 5.4.

  Is it possible to define a coordinate transformation from the unit square to a general quadrilateral ? The answer is affirmative. One has to write (fig. 32)

$$\begin{cases} x = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\,\eta \\ y = \beta_1 + \beta_2\xi + \beta_3\eta + \beta_4\xi\eta \end{cases} \qquad (6.3)$$

with the conditions

$$\begin{aligned} x_1 &= x(0,0) = \alpha_1 \\ x_2 &= x(1,0) = \alpha_1 + \alpha_2 \\ x_3 &= x(1,1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ x_4 &= x(0,1) = \alpha_1 + \alpha_3 \end{aligned}$$

from which

$$\begin{aligned} \alpha_1 &= x_1 \\ \alpha_2 &= x_2 - x_1 \\ \alpha_3 &= x_4 - x_1 \\ \alpha_4 &= x_3 - \alpha_1 - \alpha_2 - \alpha_3 = x_3 - x_1 - (x_2 - x_1) - (x_4 - x_1) = x_1 - x_2 + x_3 - x_4, \end{aligned} \qquad (6.4)$$

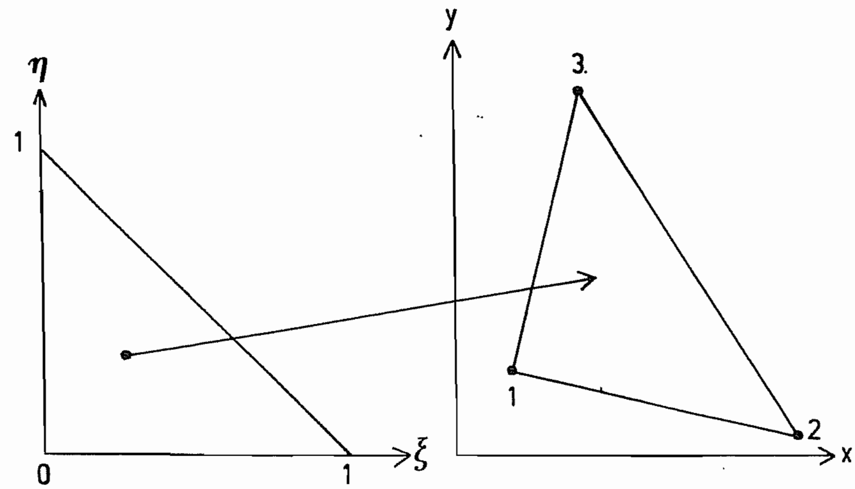


Fig. 30

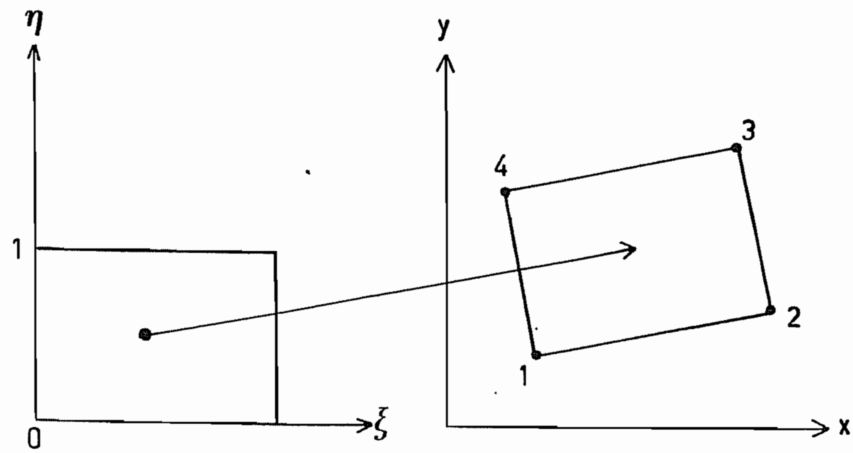


Fig. 31

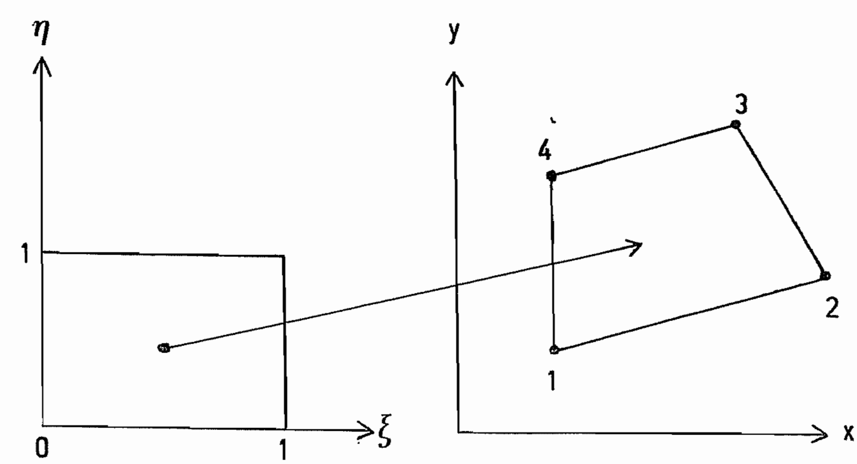


Fig. 32

and similarly for y.

Moreover, a quadratic transformation permits to obtain a curved triangle from the unit rectangular triangle (fig. 33). It is

$$\begin{aligned} x &= \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi^2 + \alpha_5\xi\eta + \alpha_6\eta^2 \\ y &= \beta_1 + \beta_2\xi + \beta_3\eta + \beta_4\xi^2 + \beta_5\xi\eta + \beta_6\eta^2 \end{aligned} \tag{6.5}$$

The curved triangle is then defined by six nodes, and the parameters α_i and β_i are obtained from the conditions

$$\begin{aligned} x_1 &= x(0,0) = \alpha_1 \\ x_2 &= x(1,0) = \alpha_1 + \alpha_2 + \alpha_4 \\ x_3 &= x(1,0) = \alpha_1 + \alpha_3 + \alpha_6 \\ x_4 &= x\left(\frac{1}{2},0\right) = \alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_4 \\ x_5 &= x\left(\frac{1}{2},\frac{1}{2}\right) = \alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + \frac{1}{4}\alpha_4 + \frac{1}{4}\alpha_5 + \frac{1}{4}\alpha_6 \\ x_6 &= x\left(0,\frac{1}{2}\right) = \alpha_1 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_6 \end{aligned}$$

and similarly for y.

In the same way, curved quadrilaterals may be obtained from the unit square by the transformation (fig. 34).

$$\begin{aligned} x &= \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi^2 + \alpha_5\xi\eta + \alpha_6\eta^2 + \alpha_7\xi^2\eta + \alpha_8\xi\eta^2 \\ y &= \beta_1 + \beta_2\xi + \beta_3\eta + \beta_4\xi^2 + \beta_5\xi\eta + \beta_6\eta^2 + \beta_7\xi^2\eta + \beta_8\xi\eta^2 \end{aligned} \tag{6.6}$$

where we let to the reader the task of defining the connection conditions.

6.2. General parametric elements

The fundamental idea is to write simultaneously the coordinates and the displacements as functions of the reference coordinates (ξ,η), it is

$$\begin{aligned} x_i &= m_x^T(\xi,\eta)\, r \\ u_i &= m_u^T(\xi,\eta)\, a \end{aligned} \tag{6.7}$$

where r and a are vectors containing parameters. At this stage, three cases are conceivable.

a) Polynomials expressing the coordinates are less rich than those that describe the displacements. The element is said *hypoparametric* or *subparametric*.

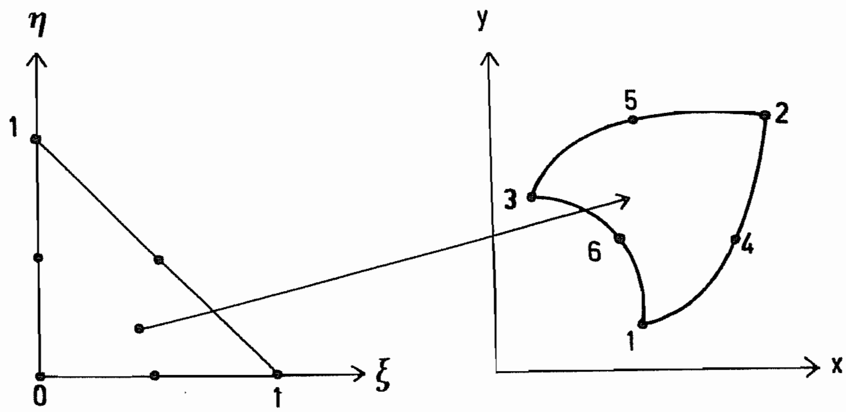


Fig. 33

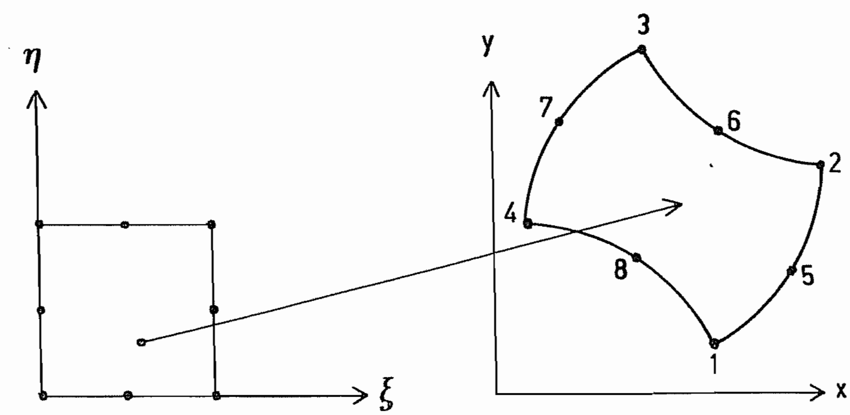


Fig. 34

b) Coordinates and displacements are polynomials of the same form. The element is then *isoparametric.*

c) Coordinates are polynommials more rich than those that express the displacements. The element is then *hyperparametric.*

Recall that a convergence condition is that rigid body motions and constant strain mode are represented exactly. This is equivalent to say that the following displacements have to be present in the element :

$$\begin{cases} u_i = 1 \\ u_i = x_1 \\ u_i = x_2 \end{cases}$$

The first condition is fulfilled when *the vector m_u contains the constant function,* which is generally satisfied.

The second implies that the displacements can be equal to the coordinates, a condition which is verified for hypo- and iso-parametric elements, but not for hyper-parametric ones. Finally, only hypo- and isoparametric elements have to be considered.

6.3. Displacements and coordinates fields in an isoparametric element.

We will restrict ourselves to the description of isoparametric elements, which are the most interesting ones and may degenerate in hypoparametrics by the proper geometrical choice. We have thus

$$\begin{aligned} x_i &= m^T(\xi_1,\xi_2)\, r\,(i) \\ u_i &= m^T(\xi_1,\xi_2)\, a\,(i) \end{aligned} \tag{6.8}$$

with r(i) and a(i) of dimension n_a. If P_1, P_2, ..., P_{n_q} are the connecting nodes, let us note

$$y(i)^T = \lfloor x_i(P_1),\dots,x_i(P_{n_q}) \rfloor \tag{6.9}$$

and

$$q(i)^T = \lfloor u_i(P_1),\dots,u_i(P_{n_q}) \rfloor \tag{6.10}$$

The connection relation is

$$y(i) = \begin{bmatrix} m^T(P_1) \\ \vdots \\ m^T(P_{n_q}) \end{bmatrix} r(i) = Cr(i) \tag{6.11}$$

and similarly,

$$q(i) = C\, a(i) \tag{6.12}$$

These connection relations are of the same form as in classical elements, and it is conceivable, although unusual in isoparametric elements, to use bubble modes for x and u. Supposing thus C invertible, one has

$$\begin{aligned} r(i) &= C^{-1}\, y(i) \\ a(i) &= C^{-1}\, q(i) \end{aligned} \tag{6.13}$$

6.4. Jacobian matrix

To compute the displacement gradients, the jacobian matrix defined by

$$J_{ij} = \begin{bmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_1}{\partial \xi_2} \\ \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} \end{bmatrix} \tag{6.14}$$

will be needed. One has

$$J^T = \left[grad_\xi x_1,\, grad_\xi x_2\right] \tag{6.15}$$

with, from expression (6.8),

$$grad_\xi x_i = grad_\xi m^T r(i) = Dr(i) \tag{6.16}$$

where it is posed

$$D = grad_\xi m^T \tag{6.16b}$$

Finally

$$J^T = [Dr(1),\, Dr(2)] \tag{6.17}$$

From this expression, it is possible to compute the jacobian det(J) and the cofactors $cof_{ij}(J)$ which are necessary to obtain the inverse jacobian matrix by

$$\left(J^{-1}\right)_{ij} = \frac{1}{\det(J)} cof_{ji}(J) \tag{6.18}$$

6.5. Displacement gradients

Displacement gradients in terms of the ξ_i's are given by

$$\text{grad}_\xi\, u_k = D\, a(k) \tag{6.19}$$

from which gradients in terms of the x_i's may be computed by

$$\text{grad}_x u_k = J^{T}\, \text{grad}_\xi u_k = J^{T}\, D\, a(k) \tag{6.20}$$

Let us define the *pseudo-strain vector*

$$e^{*} = \begin{bmatrix} grad_x u_1 \\ grad_x u_2 \end{bmatrix} \tag{6.21}$$

One has

$$e^{*} = \begin{bmatrix} J^{-T}D & 0 \\ 0 & J^{-T}D \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = Ba \tag{6.22}$$

with

$$B = \begin{bmatrix} J^{-T}D & 0 \\ 0 & J^{-T}D \end{bmatrix} \tag{6.23}$$

and

$$a^{T} = [a^{T}(1), a^{T}(2)] \tag{6.24}$$

6.6. Pseudo-Hooke matrix and energy

The same procedure will be used as with the parallelogram in section 5.4. The classical Hooke's relation

$$s = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \gamma_{12} \\ \varepsilon_{22} \end{bmatrix} \tag{6.25}$$

is modified in

$$s^* = H^* e^* \tag{6.26}$$

with

$$s^{*T} = [\sigma_{11}\ \sigma_{12}\ \sigma_{12}\ \sigma_{22}] \tag{6.27}$$

and

$$H^* = \begin{bmatrix} H_{11} & H_{12} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{22} & H_{23} \\ H_{21} & H_{22} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{32} & H_{33} \end{bmatrix} \tag{6.28}$$

so that the strain energy density may be written

$$W = \frac{1}{2} s^{*T} e^* = \frac{1}{2} e^{*T} H^* e^* = \frac{1}{2} a^T B^T H^* B a \tag{6.29}$$

The strain energy is now

$$\mathcal{U} = \int_S W dS = \int_{S_{ref}} W.\det J.d\xi_1 d\xi_2 \tag{6.30}$$

where S_{ref} is the reference element.

6.7. How to compute the strain energy ?

The problem is now to compute the integrals that are involved in (6.30). In fact, the functions $J^T D$ which are contained in the development of e^* (see 6.22) are of the form

$$\frac{cof_{ij}(J)}{\det J} \cdot D, \tag{6.31}$$

it is, *rational fractions*. An analytical integration is thus not possible, and one has to use *numerical integration*.

Any numerical integration formula is based on a set of *integration points* IP_1, IP_2, ..., IP_{nip} in the reference element and a set of corresponding integration weights IW_1, ..., IW_{nip}, and the integration scheme is of the form

$$\int_{S_{ref}} f d\xi_1 d\xi_2 \approx \sum_{k=1}^{nip} IW_k.f(IP_k) \tag{6.32}$$

The question to know what is a *suitable* integration formula will be discussed later. Let us suppose that such a formula is available. We have to compute the matrix defined by

$$\mathcal{U} = \frac{1}{2} a^T G a \tag{6.33}$$

which, from (6.29) and (6.30), is given by

$$G = \int_{S_{ref}} B^T H^* B \det J \, d\xi_1 d\xi_2 \tag{6.34}$$

To perform this computation, a loop on the nip integration points is organized as follows

- Set G = 0
- For k = 1, nip
    - compute the coordinates ξ_{1k}, ξ_{2k} of the point
    - compute matrix D_k at this point
    - compute J_k at this point
    - compute det (J_k)
    - compute $(J^{-T})_k$
    - compute B_k
    - compute $B_k^T H^* B_k$
    - set $G \leftarrow G + IW_k{}^* B_k^T H^* B_k |\det(J_k)|$

  endfor.

The only remaining task is to connect the element by the relation

$$\underbrace{\begin{bmatrix} a(1) \\ a(2) \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}}_{\overline{C}^{-1}} \underbrace{\begin{bmatrix} q(1) \\ q(2) \end{bmatrix}}_{q} \tag{6.35}$$

from which

$$\mathcal{U} = \frac{1}{2} q^T K q \tag{6.36}$$

with

$$K = \overline{C}^{-T} G \overline{C}^{-1} \tag{6.37}$$

6.8. One-dimensional integration schemes

6.8.1.- General considerations

A great majority of multidimensional integration shemes are based upon one-dimensional ones, which have thus to be recalled here. Firstly, one may consider only one reference interval]0,1[or]-1,+1[, because

- by setting x = a + (b - a) ξ, one obtains

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f\,[a+(b-a)\,\xi]\,d\xi \tag{6.38}$$

- by setting $x = \frac{a+b}{2} + \frac{b-a}{2}\xi$, one obtains

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{+1} f\left[\frac{a+b}{2} + \frac{b-a}{2}\xi\right]d\xi \tag{6.39}$$

Here, interval]0,1[will be used systematically.

The general idea of numerical integration is to replace the function f by some interpolate $\tilde{f}$ and to write

$$I(f) = \int_0^1 f\,dx \approx \int_0^1 \tilde{f}\,dx = \tilde{I}(f) \tag{6.40}$$

The interpolate is in most case a polynomial of some degree n, defined by (n+1) interpolation conditions

$$\tilde{f}(x_o) = f(x_o), \ldots, \tilde{f}(x_n) = f(x_n) \tag{6.41}$$

the x_k's being (n+1) points inside or at the boundary of the interval. The set $\{x_o,\ldots,x_n\}$ is often called the *interpolation support*. From Lagrange's interpolation formula, one has thus

$$\tilde{f}(x) = \sum_{i=o}^{n} L_i(x) f(x_i) \tag{6.42}$$

with

$$L_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)} \tag{6.43}$$

From this follows

$$\int_0^1 \tilde{f}(x)\,dx = \sum_{i=o}^{n} W_i f(x_i) \qquad (6.44)$$

where appear the integration weights

$$W_i = \int_0^1 L_i(x)\,dx \qquad (6.45)$$

This is the general procedure.

6.8.2.- Degree of an integration formula

A good integration formula should be able to integrate sufficiently complicated functions. It is customary to qualify this quality by the highest order of polynomials that are integrated exactly. This is called the *degree* of the integration formula.

Any (n+1)-point polynomial formula is *at least* of degree n because any polynomial of degree n is equal to its interpolate. But with some well-choosed supports, it is possible to obtain a higher degree. Such formulae require thus less computations for a given degree, and are therefore more economical.

6.8.3.- Numerical stability

When computing

$$\tilde{I}(f) = \int_0^1 \tilde{f}(x)\,dx = \sum_{i=o}^{n} W_i f(x_i),$$

any numerical error $\delta f(x_i)$ on the computation of the values of the function leads to an error

$$\delta \tilde{I}(f) = \sum_{i=o}^{n} W_i \delta f(x_i) \leq \sum_{i=o}^{n} |W_i|\,|\delta f(x_i)|$$

If $|\delta f(x_i)| \leq \varepsilon$, this leads to

$$|\delta \tilde{I}| \leq \varepsilon \sum_{i=o}^{n} |W_i| \qquad (6.46)$$

This result express the amplification of the error. Now, in the case where $f(x) \equiv 1$, one has

$$I(f) = \tilde{I}(f) = \sum_{i=o}^{n} W_i = \int_o^1 dx = 1 \qquad (6.47)$$

so that the sum of the weights equals 1. Therefore, *if all weights are positive,*

$$\sum_{i=o}^{n} |W_i| = \sum_{i=o}^{n} W_i = 1$$

and it follows from (6.46) that *the error on the integral never surpasses the error on the function*. In other words, *formulae with positive weights have to be preferred from a stability point of view*.

6.8.4.- Newton-Cotes formulae

Newton-Cotes formulae are characterized by uniformly distributed points, including boundaries, it is,

degree 1 : $x_o = 0, \quad x_1 = 1$

degree 2 : $x_o = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1$

degree 3 : $x_o = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1$

...

degree n : $x_i = \frac{i}{n}, \quad i = 0,\ldots,n$

The weights of the three first formulae are

degree 1 : $\frac{1}{2}, \frac{1}{2}$

degree 2 : $\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$

degree 3 : $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$

Note that these are the weights of an equally distributed side load with elements of degrees 1, 2 or 3.

Newton-Cotes formulae with an *even* number of points (n+1) are of the normal degree n. With an *odd* number of points, the degree is (n+1). This is due to the fact that any polynomial of degree (n+1) may then be written in the form

$$P_{n+1} = \alpha \prod_{i=o}^{n}(x - x_i) + R_n \,. \qquad \alpha = \text{scalar value}$$

Polynomial R_n is integrated exactly and when the number of points is odd, one has

$$\int_o^1 \prod_{i=o}^{n}(x - x_i)\, dx = 0 \quad \text{and} \quad \sum_{i=o}^{n} W_i \prod_{i=o}^{n}(x - x_i) = 0$$

so that the numerical integral of this polynomial has the correct value.

From a given number of points, Newton-Cotes formulae exhibit some negative weights and are therefore not stable.

6.8.5.- Orthogonal polynomials

For any degree n > 1, let us define the orthogonal polynomial of degree n, φ_n, as follows

- The coefficient of x^n is equal to 1
- For any polynomial P_k of degree k < n, one has

$$I(\varphi_n, P_k) = 0 \tag{6.48}$$

a) *Such polynomials exist.* It is theoretically possible to construct them by the Schmidt orthogonalization process

$$\varphi_o = 1$$

$$\varphi_1 = x - \frac{I(x, \varphi_o)}{I(\varphi_o^2)}\varphi_o$$

$$\varphi_2 = x^2 - \frac{I(x^2, \varphi_o)}{I(\varphi_o^2)}\varphi_o - \frac{I(x^2, \varphi_1)}{I(\varphi_1^2)}\varphi_1$$

and so on (practically, this process is numerically instable).

b) *These polynomials are unique.* In fact, suppose that φ_n and ψ_n are two such polynomials. Then

$$\varphi_n = x^n + P_{n-1}, \quad \psi_n = x^n + Q_{n-1}$$

where P_{n-1} and Q_{n-1} are polynomials of degree (n-1). But this implies that

$$\varphi_n - \psi_n = P_{n-1} - Q_{n-1} = R_{n-1},$$

that is a polynomial of degree (n-1). Therefore,

$$I\left[(\varphi_n-\psi_n)^2\right]=I\left((\varphi_n-\psi_n)R_{n-1}\right)=I\left(\varphi_n R_{n-1}\right)-I\left(\psi_n R_{n-1}\right)=0$$

by the definition of φ_n and ψ_n. From which follows $\varphi_n = \psi_n$.

c) *All zeros of the orthogonal polynomials are simple zeros contained in the open interval]0,1[.*

In fact, suppose it is not the case. Let us construct the function

$$g(x)=\alpha\prod{}^{*}(x-x_i)$$

where $\prod^{*}$ is the product taken on all zeros of φ_n contained in the open interval which are of odd multiplicity, each zero being only one time taken in account. With a proper choice of α, it is possible to obtain

$$g(x)\,\varphi_n(x) > 0 \text{ almost everywhere in }]0,1[. \tag{6.49}$$

Clearly, the degree of g(x) is lower than n. Therefore,

$$I(\varphi_n g) = 0$$

But from (6.49) follows

$$I(\varphi_n g) > 0$$

which is contradictory.

Exercise

Prove that $\varphi_n=\alpha\dfrac{d^n}{dx^n}\left[x^n(1-x)^n\right]$, with a proper choice of coefficient α.

6.8.6.- Gauss formulae

Let us use as an integration support the (n+1) zeros of φ_{n+1}. This defines *the Gauss formula with (n+1) points* which has very interesting properties

a) *The Gauss formula with (n+1) points is of degree (2n+1)*

To prove this, let us remark that any polynomial P_{2n+1} of degree (2n+1) may be written in the form

$$P_{2n+1} = \varphi_{n+1}\, Q_n + R_n$$

where Q_n and R_n, the quotient and the rest of the division of P_{2n+1} by φ_{n+1}, are both polynomials of degree n. Integrating, one obtains

$$I(P_{2n+1}) = I(\varphi_{n+1}\, Q_n) + I(R_n) = I(R_n) \tag{6.50}$$

because φ_{n+1} is the orthogonal polynomial of degree (n+1). Considering the numerical integral, one also obtains

$$\tilde{I}(P_{2n+1}) = \tilde{I}(\varphi_{n+1}\, Q_n) + \tilde{I}(R_n) = \tilde{I}(R_n) \tag{6.51}$$

because the integration points are precisely the zeros of φ_{n+1}. Finally,

$$\tilde{I}(R_n) = I(R_n)$$

because any polynomial integration formula with (n+1) points is at least of degree n.
So,

$$\tilde{I}(P_{2n+1}) = I(P_{2n+1})$$

which had to be proved.

b) *All weights of Gauss formulae are positive*

Les us consider the (n+1)-point formula. To point number i is associated the Lagrange polynomial $L_i(x)$, which is of degree n. From the preceeding theorem, L_i^2, which is a polynomial of degree 2n, is integrated exactly. But this implies

$$\tilde{I}(L_i^2) = W_i = I(L_i^2) > 0$$

c) Gauss points and Gauss weights are tabulated in the litterature. Note that they are generally defined on the interval]-1,+1[. The correspondance is as follows. Let x_i^* and W_i^* are Gauss points and Gauss weights on]-1,+1[, and x_i and W_i, the corresponding points and weights on]0,1[. The correspondance

$$x = \frac{1}{2} + \frac{1}{2}x^*$$

leads to

$$\int_o^1 f(x)\,dx = \frac{1}{2}\int_{-1}^{+1} f\left(\frac{1}{2}+\frac{1}{2}x^*\right) dx^*$$

whose numerical traduction is

$$\sum_i W_i f(x_i) = \frac{1}{2}\sum_i W_i^* f\left(\frac{1}{2}+\frac{1}{2}x_i^*\right)$$

from which

$$\begin{cases} x_i = \dfrac{1}{2}+\dfrac{1}{2}x_i^* \\ W_i = \dfrac{1}{2}W_i^* \end{cases} \tag{6.52}$$

6.9. Two-dimensional integration formulae

From Gauss formulae, it is possible to deduce high degree formulae in some simple two-dimensional sets.

6.9.1.- Product formula on a square

On the square]0,1[x]0,1[, one has

$$I = \int_o^1 \int_o^1 f(x,y)\,dx\,dy = \int_o^1 dx \int_o^1 f(x,y)\,dy$$

Using Gauss formula, one may write

$$I \approx \sum_i W_i \int_o^1 f(x_i, y)\,dy$$

and a second application of Gauss formula leads to

$$I \approx \sum_i W_i \sum_j W_j f(x_i, y_j) = \sum_{ij} W_i W_j f(x_i, y_j) \tag{6.53}$$

With a (n+1) x (n+1) grid, this formula is correct for each polynomial of the Q_{2n+1} type.

6.9.2.- Product formula on a general quadrilateral

The general quadrilateral may be obtained from the unit square by the transformation

$x = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta$
$y = \beta_1 + \beta_2\xi + \beta_3\eta + \beta_4\xi\eta$

with (fig. 32)

$$\begin{cases} \alpha_1 = x_1, \alpha_2 = x_2 - x_1, \alpha_3 = x_4 - x_1, \alpha_4 = x_1 - x_2 + x_3 - x_4 \\ \beta_1 = y_1, \beta_2 = y_2 - y_1, \beta_3 = y_4 - y_1, \beta_4 = y_1 - y_2 + y_3 - y_4 \end{cases} \tag{6.54}$$

The jacobian of this transformation is

$$\det J = \begin{vmatrix} \alpha_2 + \alpha_4\eta & \alpha_3 + \alpha_4\xi \\ \beta_2 + \beta_4\eta & \beta_3 + \beta_4\xi \end{vmatrix} = (\alpha_2\beta_3 - \alpha_3\beta_2) + (\alpha_4\beta_2 - \beta_4\alpha_2)\,\xi + (\alpha_4\beta_3 - \alpha_3\beta_4)\,\eta \tag{6.55}$$

One has then

$$I = \int_{quadr} f(x,y)\,dx\,dy = \int_{square} f\left[x\,(\xi,\eta), y\,(\xi,\eta)\right] \det J\, d\,\xi\, d\,\eta$$

and the numerical formula is

$$\tilde{I} = \sum_{ij} W_i W_j\, f\left[x(\xi_i,\eta_j), y(\xi_i,\eta_j)\right] dtm\, J\,(\xi_i,\eta_j) \tag{6.56}$$

Due to the presence of the jacobian which is a first degree polynomial, this formula is only exact for polynomials of Q_{2n} type (one degree is loosed).

6.9.3.- Gauss-Radau formula on a triangle

A formula for the triangle may be obtained by setting in the preceeding one

$x_4 = x_1$ and $y_4 = y$.

So, in (6.54)

$$\begin{cases} \alpha_1 = x_1, \alpha_2 = x_2 - x_1, \alpha_3 = 0, \alpha_4 = x_3 - x_2 \\ \beta_1 = y_1, \beta_2 = y_2 - y_1, \beta_3 = 0, \beta_4 = y_3 - y_2 \end{cases} \tag{6.57}$$

The jacobian is now

$$\det J = (\alpha_4\beta_2 - \beta_4\alpha_2)\,\xi \tag{6.58}$$

The formula is formally identical to (6.56). This formula, of degree 2n, exhibits a very aesthetic distribution of points (fig. 35), but it works.

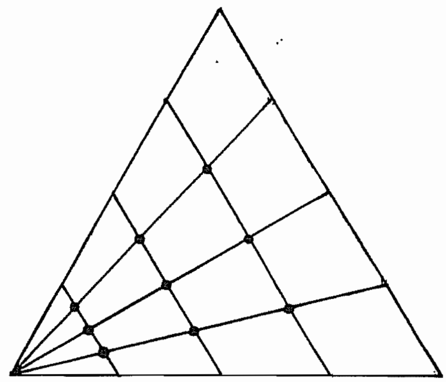


Fig. 35 : The a-esthetic distribution of Gauss-Radau points

6.9.4.- Other formulae

A lot of other formulae may be found in specialized books. Let us citate, on a triangle of area S,

a) the one-point formula (barycenter, weight = S), which is of degree 1
b) using the 3 midsides, with weight S/3, one obtains a formula of degree 2
c) the same degree is obtained with the points at 1/3 of the medians, starting from the vertices, each with a S/3 weight.

Such formulae may be verified by trying to integrate 1, x, x^2, xy, y^2, ... on a rectangular triangle.

6.10. What formula to choose for finite elements ?

6.10.1.- Introduction

Exact integration of the stiffness matrix is not possible in the case of isoparametric elements, because the functions that have to be integrated are rational fractions. Some integration error is thus unavoidable.

At a first glance, the best choice would be to use very high degree formulae. But this leads to a very expensive computation. The question is thus to define the *necessary* degree, it is the least degree that does not destroy the results. This question has been investigated by IRONS [18,19]. There are fundamentally two conditions, the first one referring to *consistency* and the second one, to *stability*.

6.10.2.- Consistency condition

The idea of this condition is as follows. When the mesh is refined, stresses in an element tend to be constant. To ensure convergence, it is thus necessary that for constant stresses, the energy variation is calculated exactly. This conditions writes

$$\delta \mathcal{U} = \tilde{I}(s^T \delta e) = I(s^T \delta e) \text{ for any constant stress}$$

or, equivalently,

$$\tilde{I}(\delta e) = I(\delta e)$$

In other words, *strains have to be integrated exactly*. This condition is completely general and applies to any element.

In the particular case of isoparametric elements, note that strains are certainly exactly integrated if vector e^* defined in (6.21) is. Now,

$$\frac{\partial u_k}{\partial x_i} = (J^{-1})_{ji} \frac{\partial u_k}{\partial \xi_j} = \frac{1}{\det J} \cdot cof_{ij}(J) \cdot \frac{\partial u_k}{\partial \xi_j}$$

and

$$\int_{element} \frac{\partial u_k}{\partial x_i} dS = \int_{ref\ elt} \frac{1}{\det J} \cdot cof_{ij}(J) \cdot \frac{\partial u_k}{\partial \xi_j} \cdot \det J \, d\xi_1 \, d\xi_2$$

$$= \int_{ref\ elt} cof_{ij}(J) \frac{\partial u_k}{\partial \xi_j} d\xi_1 \, d\xi_2 \tag{6.59}$$

Now, displacements and coordinates are of the same form, so that integral (6.59) is exactly evaluated if and only if it is the case of

$$\int_{ref\ elt} cof_{ij}(J) \frac{\partial x_k}{\partial \xi_j} d\xi_1 d\xi_2 = \int_{ref\ elt} \delta_{ik} \det J . d\xi_1 d\xi_2 \tag{6.60}$$

So, the integration formula has to be able to exactly compute the integral of the jacobian, that is, the measure of the element (area or in three dimensions, volume).

6.10.3.- Stability condition

The stability condition consists to require that *the strain energy is positive* for each displacement that is not a rigid body motion. Let

- n_q be the number of connectors
- n_r be the number of rigid body modes (3 in plane problems, 6 in spatial problems)
- n_{ip} the number of integration points
- n_ε the number of strains (3 in plane problems, 6 in spatial problems).

Then, *if all integration weights are positive*, vanishing of the calculated energy requires $\varepsilon_{ij} = 0$ at each integration points, $n_{ip} * n_\varepsilon$ conditions. The number of these conditions has to be greater than the number of straining modes, that is (n_q - n_r). So, a necessary condition for the positive definition is

$$n_{ip} * n_\varepsilon \geq n_q - n_r \tag{6.61}$$

If not verified, there exists *at least*

$$n_s = n_q - n_r - n_{ip} * n_\varepsilon \tag{6.62}$$

spurious kinematical modes.

Note the crucial role of the fact that integration weights are positive. Negative integration weights could also produce calculated strain energies that are *negative* ! Therefore, the use of positive weights formulae has to be considered as an absolute necessity.

6.10.4.- A celebrated example of spurious kinematical modes

Let us consider a rectangular element in plane strain, with displacements of the form

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2 \\ v &= \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} x^2 + \alpha_{13} xy + \alpha_{14} y^2 + \alpha_{15} x^2 y + \alpha_{16} xy^2 \end{aligned} \tag{6.63}$$

and using an integration formula with 2 x 2 Gauss points.

Displacement derivatives are given by

$$\frac{\partial u}{\partial x} = \alpha_2 + 2\alpha_4 x + \alpha_5 y + 2\alpha_7 xy + \alpha_8 y^2$$

$$\frac{\partial u}{\partial y} = \alpha_3 + \alpha_5 x + 2\alpha_6 y + \alpha_7 x^2 + 2\alpha_8 xy$$

and similarly for v. These are second degree polynomials. The Gauss formula with 2 = (1+1) points is able to integrate exactly polynomials up to degree 2*1+1=3, so that the consistency condition is verified. What about stability ? One has

$$n_q - n_r = 16 - 3 = 13$$
$$n_{ip} * n_\varepsilon = 4 * 3 = 12$$

from which, by (6.62), there is at least one spurious kinematical mode. It is relatively easy to obtain this mode analytically. For this purpose, one may, without loss of generality, consider the case of a square element and suppose that the integration points are x = ± 1, y = ± 1. We have to found a strain mode that vanishes at these points. Let us try a solution of the form

$$\gamma_{xy} = 0$$
$$\varepsilon_{xx} = \left(y^2 - 1\right) \frac{\partial}{\partial x} f(x,y)$$
$$\varepsilon_{yy} = \left(x^2 - 1\right) \frac{\partial}{\partial y} g(x,y)$$

which visibly vanishes at Gauss points. One has

$$u = \left(y^2 - 1\right) f(x,y), \qquad \frac{\partial u}{\partial y} = 2yf(x,y)$$
$$v = \left(x^2 - 1\right) g(x,y), \qquad \frac{\partial v}{\partial x} = 2xg(x,y)$$

and the condition $\gamma_{xy} = 0$, is verified by setting

$$f(x,y) = x, \quad g(x,y) = -y,$$

so that

$$u = x(y^2 - 1), \quad v = -y(x^2 - 1),$$

a solution which is effectively of the form (6.63). This displacement field, which is represented on fig. 36, is known as *hourglass mode*.

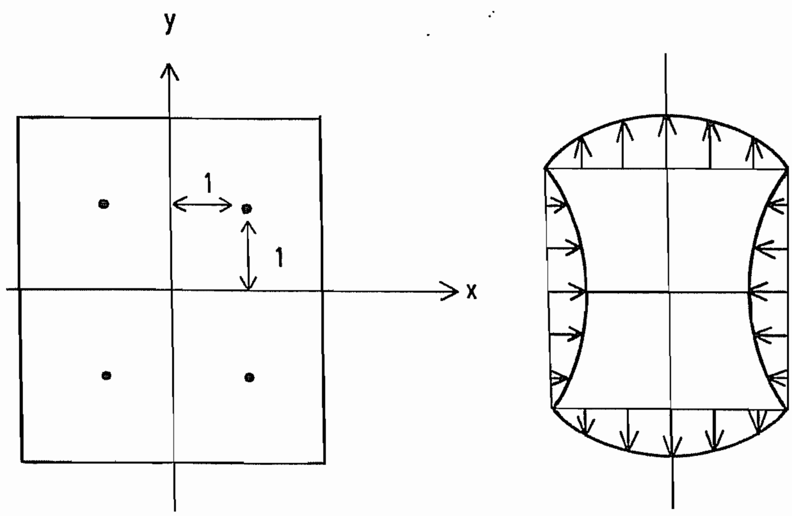


Fig. 36 : Hourglass mode

CHAPTER 7

FINITE ELEMENTS FOR PLATE BENDING

7.1. Plate theories [23]

There are several plate theories differing by their hypotheses. However, all plate theories have in common the a priori assumption

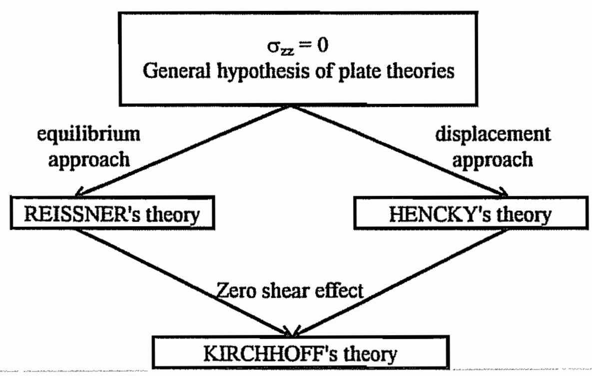
$$\sigma_{zz} = 0 \qquad (7.1)$$

Bending being characterized by antisymmetry of the strain state in terms of z, two options are common

- supposing that u and v are proportional to z and w independant of z. This is HENCKY's theory [20], which in the frame of the preliminary hypothesis (7.1), may be considered as a displacement model;

- supposing that σ_{xx}, σ_{xy}, σ_{yy} are proportional to z and verify equilibrium equations. This leads to a true equilibrium theory, due to REISSNER [21,22].

Both approaches include shear deformations, which are known to be negligibly small for very thin plates. We will thus refer to them as moderate thickness plate theories. The thin plate theory, which was developed earlier by KIRCHHOFF [24], a priori supposes that shear strains vanish. Kirchhoff's theory may be deduced from Hencky's or Reissner's one, by setting shear strains equal to zero. This situation is illustrated by the following graph.



Different theories lead to different finite elements. In this way, moderate thickness shell elements may be conceived as an alternative to thin shell elements which are by far more difficult to develop. For this reason, the following text will first develop Hencky's theory and then, reduce it to Kirchhoff's theory.

7.2. Hencky's theory

7.2.1.- General equations

The starting point is Hellinger-Reissner's principle in which the fundamental plate assumption $\sigma_{zz} = 0$ is set,

$$\int_V [\ \sigma_{xx}\frac{\partial u}{\partial x}+\sigma_{yy}\frac{\partial v}{\partial y}+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{xy}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$
$$-\frac{1}{2E}\left(\sigma_{xx}^2+\sigma_{yy}^2-2\nu\sigma_{xx}\sigma_{yy}\right)-\frac{1}{2G}\tau_{xy}^2-\frac{1}{2G}\tau_{xz}^2-\frac{1}{2G}\tau_{yz}^2\ ]\,dV+\mathcal{P}\ stat. \tag{7.2}$$

The absence of σ_{zz} in this principle suppress any condition on $\frac{\partial w}{\partial z}$.

Therefore, we are able to impose

$$w = w(x,y) \tag{7.3}$$

Concerning displacements u and v, the following assumption will be made

$$u = z\,\alpha(x,y), \quad v = z\,\beta(x,y) \tag{7.4}$$

This leads to

$$\frac{\partial u}{\partial x}=z\frac{\partial \alpha}{\partial x},\quad \frac{\partial v}{\partial y}=z\frac{\partial \beta}{\partial y},\quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=z\left(\frac{\partial \beta}{\partial y}+\frac{\partial \alpha}{\partial x}\right)$$

and

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\alpha+\frac{\partial w}{\partial x},\quad \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\beta+\frac{\partial w}{\partial y}$$

Furthermore, the applied load will be supposed to be a normal pressure p, whose energy is

$$\mathcal{P} = -\int_S pw\,dS \tag{7.5}$$

Let us first vary the stresses. One obtains the following stress-strain relations

$$\begin{aligned}
\delta\sigma_{xx} &\rightarrow z\frac{\partial\alpha}{\partial x} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \\
\delta\sigma_{yy} &\rightarrow z\frac{\partial\beta}{\partial y} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) \\
\delta\tau_{xy} &\rightarrow z\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right) = \frac{1}{G}\tau_{xy} \\
\delta\tau_{xz} &\rightarrow \left(\alpha + \frac{\partial w}{\partial x}\right) = \frac{1}{G}\tau_{xz} \\
\delta\tau_{yz} &\rightarrow \left(\beta + \frac{\partial w}{\partial y}\right) = \frac{1}{G}\tau_{yz}
\end{aligned} \tag{7.6}$$

Equilibrium conditions are then obtained by varying the displacements. Here, due allowance has to be made to the fact that α,β,w are functions of x and y only. If t is the thickness of the plate, the weak form of equilibrium is

$$\int_S \left[M_{xx}\frac{\partial\delta\alpha}{\partial x} + M_{yy}\frac{\partial\delta\beta}{\partial y} + M_{xy}\left(\frac{\partial\delta\alpha}{\partial y} + \frac{\partial\delta\beta}{\partial x}\right) + T_x\left(\delta\alpha + \frac{\partial\delta w}{\partial x}\right) + T_y\left(\partial\beta + \frac{\partial\delta w}{\partial y}\right) - p\delta w \right] dS = 0 \tag{7.7}$$

where appear the following resultants

$$\begin{aligned}
M_{xx} &= \int_{-t/2}^{t/2} \sigma_{xx} z \, dz \\
M_{yy} &= \int_{-t/2}^{t/2} \sigma_{yy} z \, dz \\
M_{xy} &= \int_{-t/2}^{t/2} \sigma_{xy} z \, dz \\
T_x &= \int_{-t/2}^{t/2} \tau_{xz} dz \\
T_y &= \int_{-t/2}^{t/2} \tau_{yz} dz
\end{aligned} \tag{7.8}$$

The equilibrium equations are thus

$$\delta\alpha \rightarrow -\frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{xy}}{\partial y} + T_x = 0$$

$$\delta\beta \rightarrow -\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_{yy}}{\partial y} + T_y = 0$$

$$\delta w \rightarrow -\frac{\partial T_x}{\partial x} - \frac{\partial T_y}{\partial y} - p = 0 \tag{7.9}$$

Derivating the equations, we supposed α = β = w = 0 on the boundary. Other cases will be investigated later. To complete our analysis, direct relations have to be written between resultants (7.8) and strains. For this, let us integrate relations (7.6). The following results are obtained

$$\frac{t^3}{12}\frac{\partial \alpha}{\partial x} = \frac{1}{E}\left(M_{xx} - \nu M_{yy}\right)$$

$$\frac{t^3}{12}\frac{\partial \beta}{\partial y} = \frac{1}{E}\left(M_{yy} - \nu M_{xx}\right)$$

$$\frac{t^3}{12}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right) = \frac{1}{G}M_{xy}$$

$$t\left(\alpha + \frac{\partial w}{\partial x}\right) = \frac{1}{G}T_x$$

$$t\left(\beta + \frac{\partial w}{\partial y}\right) = \frac{1}{G}T_y \tag{7.10}$$

and admit the following inverse expressions,

$$M_{xx} = \frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial \alpha}{\partial x} + \nu\frac{\partial \beta}{\partial y}\right)$$

$$M_{yy} = \frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial \beta}{\partial y} + \nu\frac{\partial \alpha}{\partial x}\right)$$

$$M_{xy} = \frac{Gt^3}{12}\left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right)$$

$$T_x = Gt\left(\alpha + \frac{\partial w}{\partial x}\right)$$

$$T_y = Gt\left(\beta + \frac{\partial w}{\partial y}\right) \tag{7.11}$$

It is also possible to write the complementary energy in terms of the resultants, as from (7.6)

$$\sigma_{xx} = \frac{E}{1-\nu^2} z \left(\frac{\partial \alpha}{\partial x} + \nu \frac{\partial \beta}{\partial y} \right) = \frac{12z}{t^3} M_{xx}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2} z \left(\frac{\partial \beta}{\partial y} + \nu \frac{\partial \alpha}{\partial x} \right) = \frac{12z}{t^3} M_{yy}$$

$$\tau_{xy} = Gz \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right) = \frac{12z}{t^3} M_{xy}$$

$$\tau_{xz} = G \left(\alpha + \frac{\partial w}{\partial x} \right) = \frac{T_x}{t}$$

$$\tau_{yz} = G \left(\beta + \frac{\partial w}{\partial y} \right) = \frac{T_y}{t} \tag{7.12}$$

From this follows

$$\int_{-t/2}^{t/2} \frac{1}{2E} \left[\sigma_{xx}^2 + \sigma_{yy}^2 - 2\nu \sigma_{xx} \sigma_{yy} \right] dz + \int_{-t/2}^{t/2} \frac{\tau_{xy}^2}{2G} dz + \int_{-t/2}^{t/2} \frac{\tau_{xz}^2 + \tau_{yz}^2}{2G} dz =$$

$$\frac{1}{2} \left[\frac{12}{Et^3} \left(M_{xx}^2 + M_{yy}^2 - 2\nu M_{xx} M_{yy} \right) + \frac{12}{Gt^3} M_{xy}^2 + \frac{1}{Gt} \left(T_x^2 + T_y^2 \right) \right] \tag{7.13}$$

This result leads to the following form of Hellinger-Reissner's principle.

$$\int_S \left[M_{xx} \frac{\partial \alpha}{\partial x} + M_{yy} \frac{\partial \beta}{\partial y} + M_{xy} \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right) + T_x \left(\alpha + \frac{\partial w}{\partial x} \right) + T_y \left(\beta + \frac{\partial w}{\partial y} \right) \right]$$

$$- \frac{1}{2} \left[\frac{12}{Et^3} \left(M_{xx}^2 + M_{yy}^2 - 2\nu M_{xx} M_{yy} \right) + \frac{12}{Gt^3} M_{xy}^2 + \frac{1}{Gt} \left(T_x^2 + T_y^2 \right) \right] dS - \int_S pw dS \; stat \tag{7.14}$$

whose variations may be verified to lead to equations (7.9) and (7.11).

7.2.2.- Boundary conditions with Hencky's plates

Let us suppose that on a given part $\mathscr{C}_1$ of the boundary, w = α = β = 0, and that on $\mathscr{C}_2$, loads are applied. What loads are compatible with the present model ? The answer is given by the boundary terms resulting from varying w, α, β in (7.14), which are

$$\int_{S_2} \left[\left(M_{xx}n_x + M_{xy}n_y \right) \delta\alpha + \left(M_{xy}n_x + M_{yy}n_y \right) \delta\beta + \left(n_x T_x + n_y T_y \right) \delta w \right] ds$$

Therefore, it is possible to prescribe on S_2

- A moment $\overline{M}_{xn} = M_{xx}n_x + M_{xy}n_y$, associated to $\delta\alpha$
- A moment $\overline{M}_{yn} = M_{xy}n_x + M_{yy}n_y$, associated to $\delta\beta$
- A shear load $\overline{T}_n = n_x T_x + n_y T_y$, associated to δw

7.2.3.- Strain energy in Hencky's theory

Replacing in (7.14) M_{xx}, M_{xy}, M_{yy}, T_x, T_y by their values (7.11), the following total energy principle is obtained.

$$\frac{1}{2}\int_S \left\{ \frac{Et^3}{1-\nu^2}\left[\left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2 + 2\nu\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial y} \right] + \frac{Gt^3}{12}\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right)^2 + Gt\left(\alpha + \frac{\partial w}{\partial x}\right)^2 + Gt\left(\beta + \frac{\partial w}{\partial y}\right)^2 \right\} dS + \mathcal{P}$$

stat. (7.15)

7.2.4.- Strain energy in Reissner's theory

Reissner's therory will not be developped here. For the interested reader, see SANDER [23]. The only final difference is the replacement of the factor (Gt) in shear terms by a somewhat lower one, it is $\left(\frac{5}{6}Gt\right)$. Otherwise, (7.14) and (7.15) remain valid.

7.3. Kirchhoff's theory

7.3.1.- Introduction of the so-called Kirchhoff conditions

For a given pressure field p, the equilibrium equations (7.9) implie

$$T_x = O(pL),\ T_y = O(pL) \tag{7.16}$$

where L is a characteristic length of the plate. The moments M_{xx}, M_{xy}, M_{yy} are then of order

$$M_{\alpha\beta} = O(TL) = O(pL^2) \tag{7.17}$$

From these orders of magnitude follow

$$\frac{M_{\alpha\beta}^2}{Et^3} = O\left(\frac{p^2L^4}{Et^3}\right)$$

$$\frac{T_\alpha^2}{Gt} = O\left(\frac{p^2L^2}{Gt}\right)$$

and, for a true plate (not a sandwich one!) where G/E = O(1), one observes that

$$\frac{\dfrac{T_\alpha^2}{Gt}}{\dfrac{M_{\alpha\beta}^2}{Et^3}} = O\left(\frac{t^2}{L^2}\right) \tag{7.18}$$

This leads to the conclusion that *when the plate is very thin*, that is to say $t^2/L^2 \ll 1$, *the energy due to shear loads is negligibly small*, as compared to the bending energy.

Suppressing these small terms in (7.14) leads to

$$\int_S \left[M_{xx}\frac{\partial\alpha}{\partial x} + M_{yy}\frac{\partial\beta}{\partial y} + M_{xy}\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right) + T_x\left(\alpha + \frac{\partial w}{\partial x}\right) + T_y\left(\beta + \frac{\partial w}{\partial y}\right)\right]$$

$$-\frac{1}{2}\left[\frac{12}{Et^3}\left(M_{xx}^2 + M_{yy}^2 - 2\nu M_{xx}M_{yy}\right) + \frac{12}{Gt^3}M_{xy}^2\right] dS - \int_S pwdS\ stat \tag{7.19}$$

where shear loads T_x and T_y appear as Lagrange multipliers associated to the following conditions

$$\delta T_x \rightarrow \alpha + \frac{\partial w}{\partial x} = 0$$

$$\delta T_y \rightarrow \beta + \frac{\partial w}{\partial y} = 0 \tag{7.20}$$

which are known as *Kirchhoff's conditions*. It has to be realized that these conditions express the vanishing of shear strains, not of shear loads! Shear loads remain different from zero and equilibrium equations (7.9) continue to be valid, as it is easy to verify from (7.19).

7.3.2.- Elimination of α and β

Conditions (7.20) allow the elimination of α and β in (7.19), which leads to

$$\int_S \left\{ -M_{xx}\frac{\partial^2 w}{\partial x^2} - M_{yy}\frac{\partial^2 w}{\partial y^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} - \frac{1}{2}\left[\frac{12}{Et^3}\left(M_{xx}^2 + M_{yy}^2 - 2\nu M_{xx}M_{yy}\right) + \frac{12}{Gt^3}M_{xy}^2\right]\right\} dS$$
$$-\int_S pwdS \ stat \tag{7.21}$$

Varying the moments now leads to the following relations

$$-\frac{\partial^2 w}{\partial x^2} = \frac{12}{Et^3}\left(M_{xx} - \nu M_{yy}\right)$$

$$-\frac{\partial^2 w}{\partial y^2} = \frac{12}{Et^3}\left(M_{yy} - \nu M_{xx}\right)$$

$$-2\frac{\partial^2 w}{\partial x \partial y} = \frac{12}{Gt^3}M_{xy}$$

which may be set in the inverse form

$$M_{xx} = \frac{Et^3}{12(1-\nu^2)}\left(\chi_{xx} + \nu\chi_{yy}\right)$$

$$M_{yy} = \frac{Et^3}{12(1-\nu^2)}\left(\chi_{yy} + \nu\chi_{xx}\right)$$

$$M_{xy} = \frac{Gt^3}{12}\chi_{xy} \tag{7.22}$$

where the following notations are used

$$\chi_{xx} = -\frac{\partial^2 w}{\partial x^2}$$

$$\chi_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$\chi_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \tag{7.23}$$

The equilibrium equation is obtained by varying the displacement w in (7.21). The result is

$$-\frac{\partial^2 M_{xx}}{\partial x^2}-2\frac{\partial^2 M_{xy}}{\partial x\,\partial y}-\frac{\partial^2 M_{yy}}{\partial y^2}-p=0 \qquad (7.24)$$

and is nothing than a combination of the three equations (7.9) where shear loads disappear.

7.3.3.- Boundary conditions in Kirchhoff's theory

Here lies the only true difficulty of the present theory. Due to conditions (7.20), it is no more possible to prescribe independently w, α and β on the boundary. In fact, when giving w on a connex part of the boundary, one implicitly gives its tangential derivative, and the only derivative that remains independant is the normal one, ∂w/∂n. So, *on S_1, only w and ∂w/∂n may be prescribed independently.*

In the same way, on S_2, the only loads that may be prescribed are the multiplicators of δw and $\frac{\partial \delta w}{\partial n}$. A first integration by parts gives

$$\int_S \left(-M_{xx}\frac{\partial^2 \delta w}{\partial x^2}-M_{yy}\frac{\partial^2 \delta w}{\partial y^2}-2M_{xy}\frac{\partial^2 \delta w}{\partial x\partial y}\right)dS$$

$$=-\int_{\mathscr{C}_2}\left(M_{xx}n_x\frac{\partial \delta w}{\partial x}+M_{yy}n_y\frac{\partial \delta w}{\partial y}+M_{xy}n_x\frac{\partial \delta w}{\partial y}+M_{xy}n_y\frac{\partial \delta w}{\partial x}\right)ds$$

$$+\int_S\left(\frac{\partial M_{xx}}{\partial x}\frac{\partial \delta w}{\partial x}+\frac{\partial M_{yy}}{\partial y}\frac{\partial \delta w}{\partial y}+\frac{\partial M_{xy}}{\partial x}\frac{\partial \delta w}{\partial y}+\frac{\partial M_{xy}}{\partial y}\frac{\partial \delta w}{\partial x}\right)dS$$

$$=-\int_{\mathscr{C}_2}\left[\left(n_xM_{xx}+n_yM_{xy}\right)\frac{\partial \delta w}{\partial x}+\left(n_xM_{xy}+n_yM_{yy}\right)\frac{\partial \delta w}{\partial y}\right]ds+\int_S\left(T_x\frac{\partial \delta w}{\partial x}+T_y\frac{\partial \delta w}{\partial y}\right)dS$$

where account has been taken of the definitions (7.9) of shear loads.
Now,

$$\int_S\left(T_x\frac{\partial \delta w}{\partial x}+T_y\frac{\partial \delta x}{\partial y}\right)dS=\int_{\mathscr{C}_2}T_n\delta w ds-\int_S\left(\frac{\partial T_x}{\partial x}+\frac{\partial T_y}{\partial y}\right)\delta w dS$$

where

$$T_n = n_x\,T_x + n_y\,T_y \qquad (7.25)$$

The last term will be equilibrated by the pressure. Terms on $\mathscr{C}_2$ are thus

$$\int_{\mathscr{C}_2} \left[T_n \delta w - \left(n_x M_{xx} + n_y M_{xy}\right)\frac{\partial \delta w}{\partial x} - \left(n_x M_{xy} + n_y M_{yy}\right)\frac{\partial \delta w}{\partial y}\right] ds \qquad (7.26)$$

On the boundary, let **n** be the exterior unit normal and **t** the unit tangent vector oriented in such a manner that (**n**, **t**, $\mathbf{e}_z$) form a dextrorsum referential. One may write

$$\frac{\partial \delta w}{\partial x} = \frac{\partial \delta w}{\partial n} n_x + \frac{\partial \delta w}{\partial t} t_x$$

$$\frac{\partial \delta w}{\partial y} = \frac{\partial \delta w}{\partial n} n_y + \frac{\partial \delta w}{\partial t} t_y$$

so that

$$\left(n_x M_{xx} + n_y M_{xy}\right)\frac{\partial \delta w}{\partial x} = \left(n_x^2 M_{xx} + n_x n_y M_{xy}\right)\frac{\partial \delta w}{\partial n} + \left(n_x t_x M_{xx} + n_y t_x M_{xy}\right)\frac{\partial \delta w}{\partial t}$$

$$\left(n_x M_{xy} + n_y M_{yy}\right)\frac{\partial \delta w}{\partial y} = \left(n_x n_y M_{xy} + n_y^2 M_{yy}\right)\frac{\partial \delta w}{\partial n} + \left(n_x t_y M_{xy} + n_y t_y M_{yy}\right)\frac{\partial \delta w}{\partial t}$$

and the sum is equal to

$$M_n \frac{\partial \delta w}{\partial n} + M_{nt} \frac{\partial \delta w}{\partial n}$$

with

$$M_n = M_{xx} n_x^2 + 2 M_{xy} n_x n_y + M_{yy} n_y^2 \text{ (normal moment)}$$

$$M_{nt} = M_{xx} n_x t_x + M_{xy}(n_x t_y + n_y t_x) + M_{yy} n_y t_y \text{ (torsional moment)} \qquad (7.27)$$

This reduces (7.26) to

$$\int_{\mathscr{C}_2} \left(T_n \delta w - M_n \frac{\partial \delta w}{\partial n} - M_{nt} \frac{\partial \delta w}{\partial t} \right) ds$$

The boundary may be supposed to be composed (fig. 37) of a certain number of regular arcs with corners. One has

$$-\int_{\mathscr{C}_2} M_{nt} \frac{\partial \delta w}{\partial t} ds = -\sum_{arcs} \left\{ \left[M_{nt} \delta w \right]_{node\,i}^{node(i+i)} - \int_{arc} \delta w \frac{\partial M_{nt}}{\partial t} ds \right\}$$

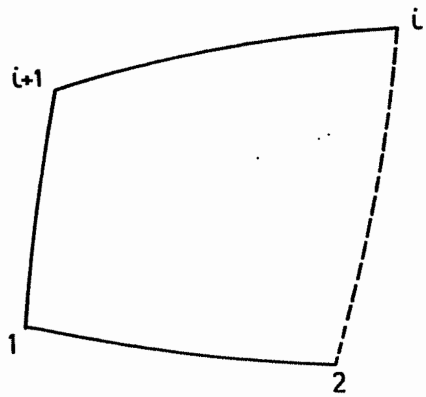

i
i+1
1
2

Fig. 37

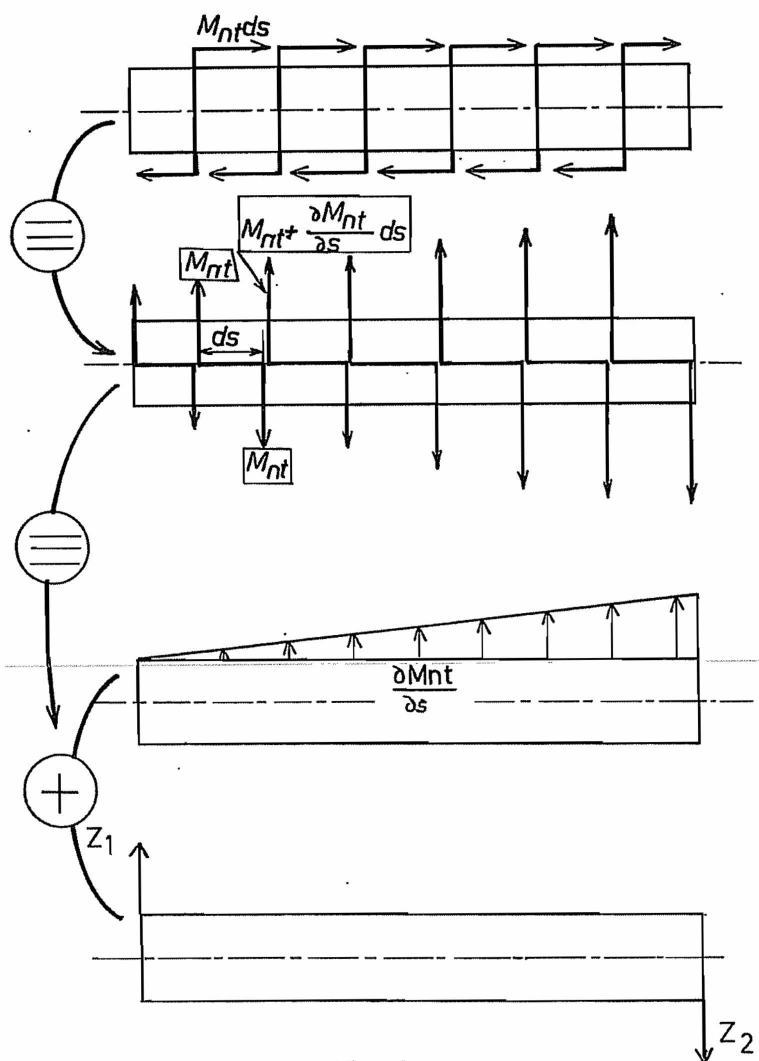

$M_{nt}ds$
$M_{nt}+\frac{\partial M_{nt}}{\partial s}ds$
M_{nt}
ds
M_{nt}
$\frac{\partial Mnt}{\partial s}$
Z_1
Z_2

Fig. 38

$$= + \sum_{\substack{corners \\ n^r_i}} \left[M_{nt}(i_+) - M_{nt}(i_-)\right]\delta w_i + \int_{\mathscr{C}_2} \delta w \frac{\partial M_{nt}}{\partial t} ds$$

so that one finally obtains

$$\int_{\mathscr{C}_2}\left[K_n \delta w - M_n \frac{\partial \delta w}{\partial n}\right] ds + \sum_{corners} Z_i \delta w_i \tag{7.28}$$

where

$$K_n = T_n + \frac{\partial M_{nt}}{\partial t} \tag{7.29}$$

is the *Kirchhoff shear load* and

$$Z_i = M_{nt}(i_+) - M_{nt}(i_-) \tag{7.30}$$

are *corner loads*. So, in Kirchhoff's theory, it is only possible to specify the normal moment, the Kirchhoff shear load and corner forces.

7.3.4.- Thomson-Tait interpretation of the boundary conditions

This situation, however obtained mathematically (and our exposition is precisely Kirchoff's one), is somewhat surprising. A physical explanation has been given by THOMSON and TAIT [25]. As represented on fig. 38, due to Kirchoff's condition, a torsional moment is equivalent to a couple of shear loads. A distribution of such couples is the equivalent to a distributed shear load equal to the derivation of M_{nt}, and two end loads.

7.3.5.- Strain energy

Eliminating the moments in (7.21) with the aid of (7.22) leads to the following expression of to total energy

$$\mathscr{E} = \frac{1}{2}\int_S \frac{Et^3}{12(1-\nu^2)} \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dS + \mathscr{P} \tag{7.31}$$

where use is made of the relation

$$G = \frac{E}{2\,(1+\nu)}$$

An equivalent form, which is often citated is

$$\mathcal{E} = \frac{1}{2}\int_S \frac{Et^3}{12(1-\nu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\,(1-\nu)\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)\right] dS + \mathcal{P} \quad (7.32)$$

7.4. On the difficulty to generate conforming thin plate elements

The strain energy of thin plates containing second order derivatives of the displacement, *C^1-connections have to be obtained.* In other words, not only w but also $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ have to be continuous.

Let us first consider a triangular element. Suppose that w, $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ are connected at the nodes (fig. 39). Then, on an interface, w is determined by 4 values, w_1, w_2, $\left(\frac{\partial w}{\partial t}\right)_1, \left(\frac{\partial w}{\partial t}\right)_2$, the two last ones being consequences of the nodal gradients. The displacement is therefore at least of degree 3. But if w is a complete 3d-degree polynomial, $\partial w/\partial n$ is of degree 2 and its connection necessitates 3 local values. The nodal values are thus not sufficient, and a supplementary value, at the midpoint of the side, is necessary. The connections are finally represented in fig. 40.

Generally speaking, a conforming triangle of degree $k \geq 3$ possesses the following connectors :

- Nodal values of w :	3
- Nodal values of derivatives :	6
- interfacial values of w :	3(k - 3)
- interfacial values of $\frac{\partial w}{\partial n}$:	3(k - 2)

This leads to a number n_q of connectors

$$n_q = 9 + 6k - 15 = 6(k - 1) \quad (7.33)$$

Now, the number of parameters of a complete polynomial of degree k is given by

$$n_a = \frac{(k+1)\,(k+2)}{2} \quad (7.34)$$

but they could exist bubbles which are not connectable. These bubbles, being polynomials that vanish with their derivatives on each side, are of the form

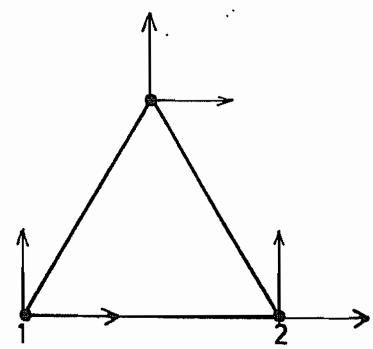


Fig. 39

● = displacement

→ = slope

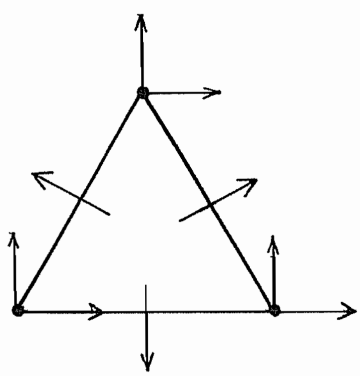


Fig. 40

$$w = C_1^2 C_2^2 C_3^2 P_{k-6} \tag{7.35}$$

where $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ are the equations of the three sides. The number of bubbles is thus

$$n_b = 0 \text{ if } k < 6$$
$$n_b = \frac{(k-5)(k-4)}{2} \text{ if } k \geq 6 \tag{7.36}$$

The following table may thus be established

k	3	4	5	6	7	8
n_a	10	15	21	28	36	45
n_b	0	0	0	1	3	6
n_a-n_b	10	15	21	27	33	39
n_q	12	18	24	30	36	42
n_s	2	3	3	3	3	3

The number of spurious kinematical modes $n_s = n_q - (n_a - n_b)$ is visibly *always positive*. For any value of $k \geq 6$ it is given by

$$n_s = 6k - 6 - \frac{1}{2}(k^2 + 3k + 2) + \frac{1}{2}(k^2 - 9k + 20) = 3\,.$$

It is therefore impossible to construct strictly conforming elements of the triangular shape with complete polynomials.

7.5. First remedy - Use of moderately thick plate elements

A first way to solve this difficulty consists to use moderate thickness plate elements. Here, w, α and β appear in the energy as first derivatives, and a C°-continuity for each field is sufficient. The normal way is to use a polynomial of degree k for w and polynomials of degree (k - 1) for α and β, which must be close to derivatives of w.

The element of degree 3 for w is represented at fig. 41. It possesses 21 connectors and 1 bubble-mode for w, in place of the 12 connectors of a Kirchhoff element. *Such elements are thus more expensive.*

Such a supplementary expense could be supported if there were no other problem. But precisely, *for very thin plates, these elements are ill-conditioned* because the stiffness combines terms O(t) and O(t^3) which are out of proportion.

Moderately thick plate elements are thus not a complete answer to the problem.

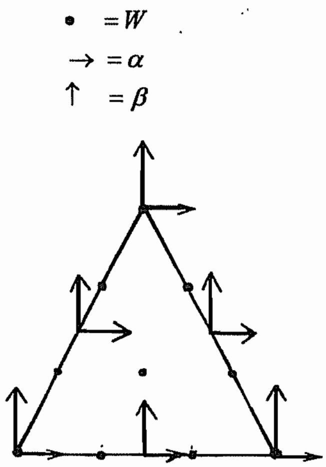


Fig. 41

7.6. Second remedy - More than conforming elements

Another possibility is to use more than conforming connections. In this way, let us citate an element which is often attributed to ARGYRIS [26] but seems to have been obtained independently by a lot of authors [17]. The idea is to connect at each node of a triangle not only w, $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ but also $\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}$.

On each side, the displacement is thus defined by two values of w, two values of $\frac{\partial w}{\partial t}$ and two values of $\frac{\partial^2 w}{\partial t^2}$, that make 6 values. W has thus to be of degree 5 at least. For $\frac{\partial w}{\partial n}$, which is then of degree 4, one has 2 values of $\frac{\partial w}{\partial n}$ and 2 values of $\frac{\partial^2 w}{\partial n \partial t}$, that is, 4 conditions only. One supplementary value of $\frac{\partial w}{\partial n}$ has thus to be added at the midsides. This leads to 3 x 6 nodal connectors and 3 midside slopes, i.e. 21 connectors. But this is precisely the number of parameters of a 5th-degree polynomial (fig. 42).

This element is clearly more than conforming, as second derivatives are connected at the nodes. This extra-conformity has no bad effect when the solution is very regular. But in the case of irregular solutions, which are common with concentrated loads or when the thickness of adjacent elements are different, more than conforming elements converge slowly.

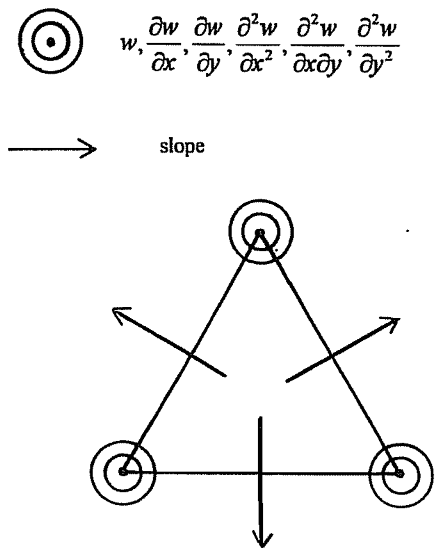


Fig. 42

7.7. Third remedy - Assembled elements

7.7.1.- FRAEIJS de VEUBEKE - SANDER quadrilateral

Suppose that on each side of a quadrilateral element, the displacement is of degree 3 and the normal slope of degree 2. The exactly conforming connection, as represented on fig. 43, requires 16 connectors. This is not possible with a unique third degree polynomial. But it becomes possible by a assembling 4 subelements in a special way.

Complete polynomials may be defined in any cartesian system, even in oblique axes. Let us choose the two diagonals as axes. These diagonals cut the quadrangle in 4 triangles (fig. 44). In triangle I, a complete third degree polynomial will be used,

$$w_I = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x2y + \alpha_9 xy^2 + \alpha_{10} y^3 \quad (7.37)$$

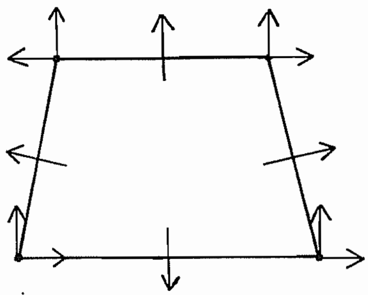


Fig. 43

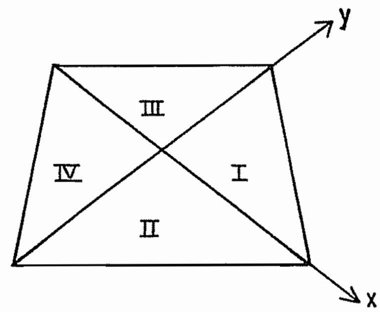


Fig. 44

The conformity condition between element I and II imposes the continuity of w and its derivatives. Consequently, if w_{II} is also a complete third degree polynomial, it is of the form

$$w_{II} = w_I + y^2 (\alpha_{11} + \alpha_{12}x + \alpha_{13}y) \tag{7.38}$$

Similarly, on triangle III, the only admissible form is

$$w_{III} = w_I + x^2 (\alpha_{14} + \alpha_{15}x + \alpha_{16}y) \tag{7.39}$$

On triangle IV, two conditions have to be fulfilled. The first one is conformity with triangle II, which imposes

$$w_{IV} = w_{II} + x^2 (\beta_1 + \beta_2 x + \beta_3 y) \tag{7.40}$$

The second condition is conformity with triangle III, from which

$$w_{IV} = w_{III} + y^2 (\gamma_1 + \gamma_2 x + \gamma_3 y) \tag{7.41}$$

But these two conditions are fulfilled by the following expression

$$w_{IV} = w_I + y^2 (\alpha_{11} + \alpha_{12}x + \alpha_{13}y) + x^2 (\alpha_{14} + \alpha_{15}x + \alpha_{16}y) \tag{7.42}$$

(and one easily verifies that this is the only solution). Finally, one has obtained a composite field which on each side is of degree 3 with 2d-degree normal slopes, and which possesses eactly 16 degrees of freedom as required. This pretty element is often referred as CQ element (Conforming Quadrilateral).

7.7.2.- Practical generation of the CQ element

Practically, it is by far more easy to use cartesian coordinates. The strategy is then as follows (fig. 44bis).

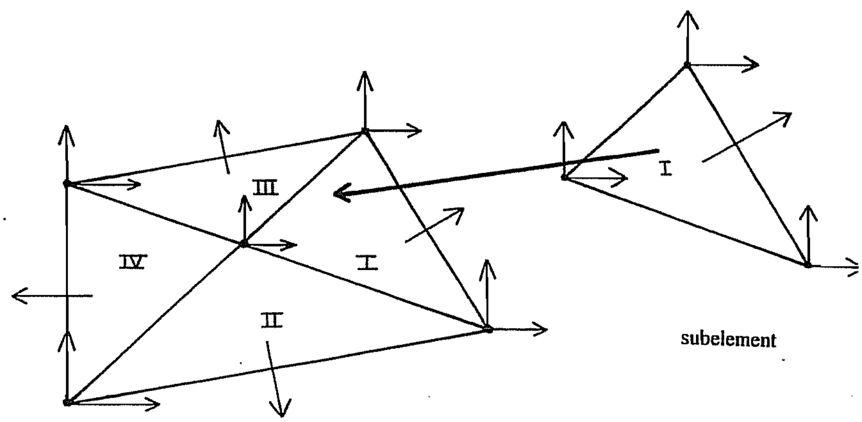


Fig. 44bis

(i) In each subelement, use a complete cubic, and as connectors, the displacement and the slopes at each node, and the normal slope at the exterior side only. This makes 10 connectors, for a 10-parameter field.

(ii) After assembling, express the three interior values w_o, w_{xo}, w_{yo} to obtain continuity of the slope on three interior interfaces, say interface I-II, interface I-III and interface III-IV. The slope continuity on the 4th interface will be automatically satisfied!

This procedure requires of course some justifications. First of all, *is the connection matrix of each subelement invertible* ? As it is a square matrix, we have only to prove that the condition $q = Ca = 0$ implies $a = 0$. Let us take the internal node as origin of the axes. The equation of the exterior interface is of the form (fig. 45).

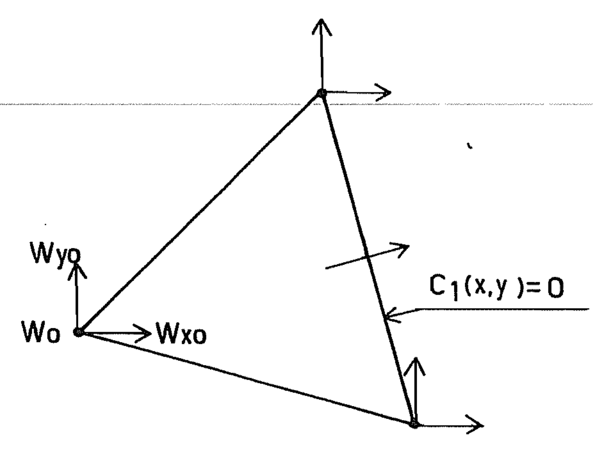


Fig. 45

$$C_1(x,y) = 1 + Ax + By = 0$$

with $A \neq 0$, $B \neq 0$. Due to the fact that all connectors on this interface vanish, the displacement is of the form

$$w = C_1^2(\alpha_1 + \alpha_2 x + \alpha_3 y)$$

We have thus at the origin

$$C_1(0,0) = 1; \quad \left(\frac{\partial C_1}{\partial x}\right)_0 = A; \quad \left(\frac{\partial C_1}{\partial y}\right)_0 = B$$

and

$$w_o = \alpha_1$$

$$w_{xo} = \left[2C_1 A(\alpha_1 + \alpha_2 x + \alpha_3 y) + C_1^2 \alpha_2\right]_o = 2A\alpha_1 + \alpha_2$$

$$w_{yo} = \left[2C_1 B(\alpha_1 + \alpha_2 x + \alpha_3 y) + C_1^2 \alpha_3\right]_o = 2B\alpha_1 + \alpha_3$$

from which

$$\alpha_1 = w_o, \quad \alpha_2 = w_{xo} - 2\,Aw_o, \quad \alpha_3 = w_{yo} - 2\,Bw_o$$

It is clear that $w_o = w_{xo} = w_{yo}$ implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$, that is $a = 0$.
So the connection matrix is a regular one.

The second question is to prove *that the connection of the midside slopes on three internal interfaces only is sufficient*. Let us suppose that this connection is made on interfaces I-II, I-III and II-IV. Reasoning in oblique axes, one has thus exact compatibility at these interfaces, so that (fig. 44)

$$w_{II} = w_I + y^2(\alpha_1 + \alpha_2 x + \alpha_3 y)$$

$$w_{III} = w_I + x^2(\alpha_4 + \alpha_5 x + \alpha_6 y)$$

$$w_{IV} = w_{II} + x^2(\alpha_7 + \alpha_8 x + \alpha_9 y) = w_I + y^2(\alpha_1 + \alpha_2 x + \alpha_3 y) + x^2(\alpha_7 + \alpha_8 x + \alpha_9 y)$$

and we have to prove that the compatibility between triangles III and IV is achieved. One has

$$w_{IV} - w_{III} = x^2\left[(\alpha_7 - \alpha_4) + (\alpha_8 - \alpha_5)\,x + (\alpha_9 - \alpha_6)\,y\right] + y^2(\alpha_1 + \alpha_2 x + \alpha_3 y)$$

and this function should vanish with its derivatives on the interface III-IV whose equation is $y = 0$. Firstly, connections at the central node and at node A implie the continuity of w and consequently of $\frac{\partial w}{\partial x}$. So,

$$(w_{II} - w_{III})_{y=0} = x^2 [(\alpha_7 - \alpha_4) + (\alpha_8 - \alpha_5) x] = 0$$

whatever x, and this implies

$$\alpha_7 = \alpha_4, \quad \alpha_8 = \alpha_5$$

We have now to consider the jump of $\frac{\partial w}{\partial y}$, which is

$$\left[\frac{\partial w_{IV}}{\partial y} - \frac{\partial w_{III}}{\partial y}\right]_{y=0} = \left[x^2(\alpha_9 - \alpha_6) + 2\alpha_1 y + 2\alpha_2 xy + 3\alpha_3 y^2\right]_{y=0}$$

$$= x^2(\alpha_9 - \alpha_6)$$

But at point A, $x = x_A$ and the slope is connected, so that

$$x_A^2(\alpha_9 - \alpha_6) = 0$$

which implies $\alpha_9 = \alpha_6$, it is to say that $\frac{\partial w}{\partial y}$ is connected.

7.7.3.- Assembled triangle of HSIEH, CLOUGH and TOCHER [29], fig. 46

This element, sometimes referred as the HCT triangle, consists to subdivide the element in three triangular subelements. The procedure is the same as described in section 7.7.2., and the three values w_o, w_{xo}, w_{yo} are choosed to obtain slope continuity on the three interior interfaces. It seems that this element, hower simpler than the CQ element, was discovered later.

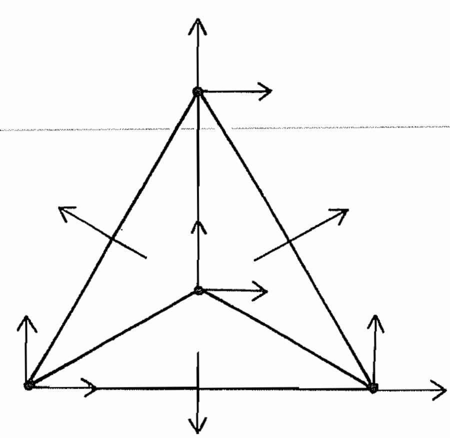


Fig. 46

7.8. Fourth remedy - Non conforming elements

The last way to overcome the difficulties encountered in plate elements consists to use non conforming elements. Very effective elements have been developed in this way. Their study will be made in chapter 8 where the general theory of nonconforming elements is given.

7.9. The singularity theorem

Confronted to very sophisticated elements as the CQ or the HCT, the reader probably will ask the following question, *is such a complication really necessary* ? The answer lies in the following singularity theorem, which says that *it is impossible to generate a conforming plate element with a displacement field which is of the C^2-class within the element.*

In fact, consider (fig. 47) a node A of a conforming plate element. Adjacent sides AB and AC will be taken as axes. Due to conformity, on AB, $\partial w/\partial y$ may be imposed to be equal to any function $\varphi(x)$. Similarly, we may suppose that on AC, $\partial w/\partial x = \psi(y)$. No correlation exists between $\varphi(x)$ and $\psi(y)$. But this implies that

- on AB, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial y}\right) = \varphi'(x)$

- on AC, $\dfrac{\partial}{\partial y}\left(\dfrac{\partial w}{\partial x}\right) = \psi'(y)$

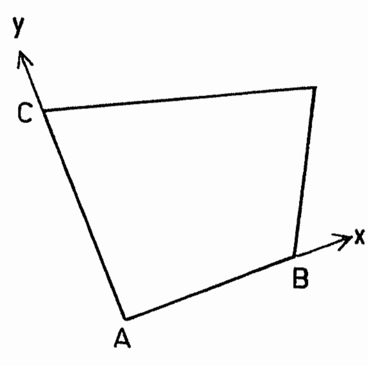


Fig. 47

and, at point A,

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \varphi'(0)$$

$$\frac{\partial y}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \psi'(0)$$

Due to the fact that φ(x) and ψ(y) are independent functions, one obtains

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\right]_A \neq \left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\right]_A$$

so that w is not of the C^2-class. This is why an element subdivision is necessary to obtain strict conformity.

CHAPTER 8

NONCONFORMING ELEMENTS

8.1. Introduction

The classical Rayleigh-Ritz code imposes conformity between elements. From this point of view, nonconforming elements may be considered as "variational crimes", as said STRANG and FIX [32]. However, due to the difficulty to ensure conformity in some cases, e.g. in plate elements, various nonconforming elements were experimented. The overall result was that *nonconforming elements may and may not converge*. The fact that some elements do converge was not explained until B. IRONS presented his now celebrated *patch test* [31] which permits to determine if convergence will or will not occur. This was the starting point of the development of a wide class of useful elements.

8.2. ADINI's element [36]

It is a rectangular plate element with the following displacement field

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \tag{8.1}$$

The connectors are the values of w, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ at the four nodes (fig. 47bis), and their number is thus twelve also.

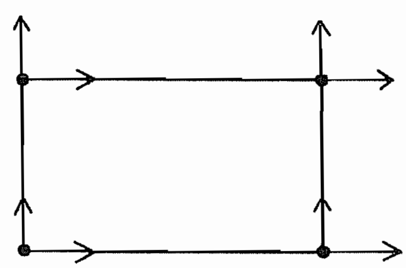


Fig. 47bis

On each side x = ct or y = ct, the displacement is a third degree polynomial, and is connected by two values of w and two values of $\frac{\partial w}{\partial t}$. Thus, the continuity of the *displacement* is guaranteed. This is not the case for the slopes. In fact,

$$\frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_4 x + \alpha_5 y + 3\alpha_7 x^2 + 2\alpha_8 xy + \alpha_9 y^2 + 3\alpha_{11} x^2 y + \alpha_{12} y^3$$

and for a constant x, this reduces to

$$\frac{\partial w}{\partial x} = (\alpha_2 + 2\alpha_4 x + 3\alpha_7 x^2) + (\alpha_5 + 2\alpha_8 x + 3\alpha_{11} x^2)y + \alpha_9 y^2 + \alpha_{12} y^3 \tag{8.2}$$

that is a third-degree polynomial of y. It is clear that the two nodal values of $\frac{\partial w}{\partial x}$ at the ends of a side (x = ct) cannot ensure a unique value of the slope. So, this element is not conforming, but *experiences show that it always converges in the rectangular shape* (not for a general quadrilateral).

8.3. WILSON's element [35]

In contrast with Adini's element which was nonconforming because exact conformity is difficult to obtain with plate elements, Wilson's element is basically a conforming element in which nonconforming modes are added to improve the results.

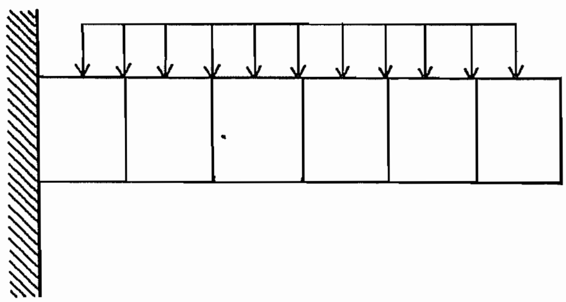


Fig. 48

The idea is as follows. When using first degree (Q_1) elements to represent a beam submitted to transverse loads (fig. 48) very bad results are obtained, because these elements cannot reproduce any curvature, so that vertical displacements are necessarily related to shear. And it is well known that with beams, shear effects represent only the small part of the deformation (this effect is now referred as shear locking). A solution will be to use second degree elements, but this leads to a greater number of degrees of freedom. Moreover, the use of interfacial degrees of freedom involves a greater sophistication of the software. For these reasons, Wilson added two internal nonconforming modes on each field, writing (fig. 49)

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \beta_1 x(x - a) + \beta_2 y(y - b) \\ v &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy + \beta_3 x(x - a) + \beta_4 y(y - b) \end{aligned} \tag{8.3}$$

The "internal" parameters βi are then condensed such as bubble modes. Surprisingly, *this element converges when rectangular*, and leads to very better results than the original Q_1 element. It does not converge for other quadrilateral shapes.

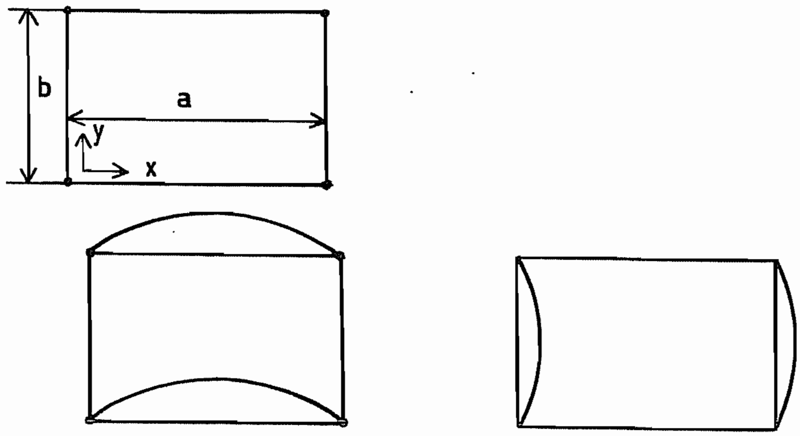


Fig. 49 : The nonconforming modes of Wilson's element

8.4. MORLEY's element

A very simple triangular plate element has been proposed by MORLEY [37], in which the displacement is approximated by a second degree polynomial,

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \tag{8.4}$$

The connectors are (fig. 50) w at the nodes, $\frac{\partial w}{\partial n}$ at the mid-sides. Here, the displacement itself is discontinuous. *This element also converges*, although exhibiting for coarse meshes an excessive flexibility.

It has however to be said that this convergence result caused no surprise because this element may be re-interpreted in the frame of the equilibrium approach.

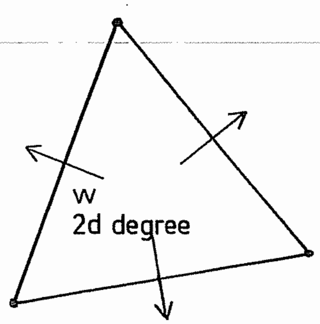


Fig. 50 : Morley's element

8.5. Conforming and nonconforming parts of displacement traces on an interface

Considering the interface between two elements (fig. 51), the traces on this interface of the displacements of each element are different. However, partial connections always exist. Let us define the *conforming part of the displacement* v_i as the simplest interpolation of the connected values. It will be noted $\tilde{v}_i$. It may be considered that $\tilde{v}_i$ is connected between the elements. In contrary, the difference

$$\Delta v_i = v_i - \tilde{v}_i \tag{8.5}$$

differs from one element to the other. It will be called *the nonconforming part of the displacement.*

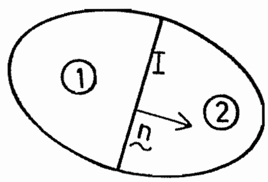


Fig. 51

Some examples will make these new concepts perfectly clear.

a) In the case of Adini's element, the displacement is continuous, so that $w = \tilde{w}$, $\Delta w = 0$. Concerning the slopes, nodal values are connected. On a given interface, say a vertical one, the slope may be expressed from (8.2)

$$\varphi_n = \beta_1 + \beta_2 y + \beta_4 y^2 + \beta_5 y^3$$

The two nodal values, corresponding to y = 0 and y = b are

$$\varphi_{n1} = \beta_1$$
$$\varphi_{n2} = \beta_1 + \beta_2 b + \beta_4 b^2 + \beta_5 b^3$$

The simplest interpolation is thus

$$\tilde{\varphi}_n = \beta_1 + \beta_2 y + \beta_4 by + \beta_5 b^2 y$$

This is the conforming part of the slope. The difference

$$\Delta\varphi_n = \varphi_n - \tilde{\varphi}_n = \beta_4(y^2 - by) + \beta_5(y^3 - b^2 y)$$

is the nonconforming part of the slope.

b) In the case of Wilson's element, one has on a vertical side, from (8.3),

$$u = \gamma_1 + \gamma_2 y + \beta_2 y\,(y - b)$$

and this displacement is connected at y = 0 and y = b. Here, the decomposition is trivial,

$$\tilde{u} = \gamma_1 + \gamma_2 y$$
$$\Delta u = \beta_2 y\,(y - b)$$

c) In the case of Morkey's element, both w and φ_n are discontinuous on the interfaces. If s is a local coordinate on an interface, varying from o to l, one has

$$w = \beta_1 + \beta_2 s + \beta_3 s^2$$

with connections at the two nodes, where the values are

$$w_1 = \beta_1$$
$$w_2 = \beta_1 + \beta_2 l + \beta_3 l^2$$

The conforming part of the displacement is

$$\tilde{w} = w_1 + \frac{s}{l} w_2 = \beta_1 + \beta_2 s + \beta_3 ls$$

and its nonconforming part is given by

$$\Delta w = w - \tilde{w} = \beta_3 (s^2 - sl)$$

Concerning the slope, its general form is

$$\varphi_n = \gamma_1 + \gamma_2 s$$

and the only connected value is

$$\varphi_{12} = \varphi_n(l/2) = \gamma_1 + \gamma_2 \frac{l}{2}$$

The conforming part of the slope is here constant,

$$\tilde{\varphi}_n = \gamma_1 + \gamma_2 \frac{l}{2}$$

and the nonconforming part is

$$\Delta\varphi_n = \varphi_n - \tilde{\varphi}_n = \gamma_2 \left(s - \frac{l}{2} \right)$$

8.6. Applied loads on interfaces

Here, the only simple way is to apply the interface loads not on the total displacements but on their conforming parts. The work on S_2 will thus be

$$\int_{S_2} \bar{t}_i \tilde{u}_i dS, \tag{8.6}$$

and not

$$\int_{S_2} \bar{t}_i u_i dS$$

As an example, a constant normal moment on a side of Adini's element will be reported in two equal values at the end nodes of the side. This is the easiest way and we will see, when discussing the patch test, that it is the only correct one.

8.7. The patch test

The question of convergence of nonconforming elements remained obscure until B. IRONS [31] proposed the *patch test*. This test will be introduced here as a necessary condition of convergence.

8.7.1.- What kind of convergence may be expected ?

Engineers are generally concerned by a stress convergence. Finite elements never guarantee a local convergence, but a *root mean square convergence*. That is to say, a finite element stress field σ_{ij}^h coming from elements of side h approximates the exact stress field σ_{ij} at an order h^r if

$$\left[\frac{1}{V}\int_V \left(\sigma_{ij}-\sigma_{ij}^h\right)\left(\sigma_{ij}-\sigma_{ij}^h\right)dV\right]^{1/2} \leq C\, h^r$$

with some constant C not depending on h. Using the L^2-norm notation,

$$\|\sigma\|_V^2 = \int_V \sigma_{ij}\sigma_{ij} dV,$$

the above condition writes

$$\left\|\sigma-\sigma^h\right\|_V \leq Ch^r V^{1/2} \tag{8.7}$$

8.7.2.- The nonconforming equilibrium equations

In a nonconforming finite element model, whose displacement variations δu_i^h will be noted v_i^h for brevity, the stresses σ_{ij}^h are obtained from the weak equilibrium conditions (e = element)

$$\sum_e \int_e \sigma_{ij}^h D_j v_i^h dV - \sum_e \int_e f_i v_i^h dV - \int_{S_2} \bar{t}_i \tilde{v}_i^h dS = 0 \tag{8.8}$$

If the displacements were conforming, these equilibrium conditions would be exact, that is, also verified by the true stress field. But with nonconforming elements, the weak equilibrium equations are perturbed, and this is why the solution may or may not converge. The ***equilibrium error*** as computed by

$$E(\sigma, v^h) = \sum_e \int_e \sigma_{ij} D_j v_i^h dV - \sum_e \int_e f_i v_i^h dV - \int_{S_2} \bar{t}_i \tilde{v}_i^h dS \tag{8.9}$$

is a measure of the disturbance due to non conformity. It may be interpreted as a consistency error of the model.

8.7.3.- Necessary condition for convergence

Let us suppose that the approximate stresses σ_{ij}^h converge to the true stress field σ_{ij} at an order r. From Schwarz-Cauchy inequality, this implies for each v^h,

$$\sum_e \int_e \left(\sigma_{ij} - \sigma_{ij}^h\right) D_j v_i^h dV \leq \left\|\sigma-\sigma^h\right\|_V \left\|Dv^h\right\|_V \leq Ch^r V^{1/2} \left\|Dv^h\right\|_V$$

where use is made of the notation

$$\left\|Dv^h\right\|_V^2 = \sum_e \int_e D_j v_i^h D_j v_i^h dV .$$

Now, from the weak equilibrium conditions, this reduces to

$$E(\sigma, v^h) \leq C\, h^r V^{1/2} \left\|Dv^h\right\|_V \tag{8.10}$$

that is a condition on the equilibrium error. It may be proved that this condition is also sufficient, but the proof will be omitted in the present text.

8.7.4.- The equilibrium error as an incompatibility work

The first term of the equilibrium error may be integrated by parts in each element (recall that v_i^h is discontinuous at interelement boundaries), leading to

$$\sum_e \int_e \sigma_{ij}^h D_j v_i^h dV = \sum_e \int_{\partial e} \sigma_{ij} n_j v_i^h dS - \sum_e \int_e v_i^h D_j \sigma_{ji} dV \tag{8.11}$$

Now, in the boundary term, v_i^h may be decomposed in conforming and nonconforming parts,

$$\sum_e \int_{\partial e} \sigma_{ij} n_j v_i^h dS = \sum_e \int_{\partial e} \sigma_{ij} n_j \tilde{v}_i^h dS + \sum_e \int_{\partial e} \sigma_{ij} n_j \Delta v_i^h dS \tag{8.12}$$

The element boundaries may be on S_1, on S_2, or interelement boundaries I, so that

$$\sum_e \int_{\partial e} \sigma_{ij} n_j \tilde{v}_i^h dS = \int_{S_1} \sigma_{ij} n_j \tilde{v}_i^h dS + \int_{S_2} \sigma_{ij} n_j \tilde{v}_i^h dS + \sum_I \int_I \left[\left(\sigma_{ij} n_j\right)_+ + \left(\sigma_{ij} n_j\right)_- \right] \tilde{v}_i^h dS \tag{8.13}$$

where on interfaces, indexes + and - are used to note the two adjacent elements. Assembling results (8.11) to (8.13) and taking in account the equilibrium equations

$$\begin{cases} D_j \sigma_{ji} + f_i = 0 \; in \; V \\ n_j \sigma_{ji} = \bar{t}_i \; on \; S_2 \\ (n_j \sigma_{ji})_+ + (n_j \sigma_{ji})_- = 0 \; on \; the \; interfaces \end{cases}$$

and the kinematical conditions $\tilde{v}_i^h = 0$ on S_1, one finally obtains

$$E\left(\sigma, v^h\right) = \sum_e \int_{\partial e} \sigma_{ij} n_j \Delta v_i^h = \mathcal{I} \tag{8.14}$$

that is the work of incompatibililty on the true stress field. This work has to converge to zero when the mesh is refined.

8.7.5.- The element patch test

How is it possible to verify this condition ? Here appears the *patch test. Suppose that for each element, the incompatibility work vanishes for each stress field* $\hat{\sigma}_{ij}$ *which is a complete polynomial of degree (r - 1),*

$$\int_{\partial e} \hat{\sigma}_{ij} n_j \Delta v_i^h dS = 0 \quad \forall \hat{\sigma}_{ij} \in P_{r-1} \tag{8.15}$$

One says that is passes a patch test of order (r - 1).

Then, supposing that σ_{ij} is of class C^r, take as $\hat{\sigma}_{ij}$ the (r-1) order Taylor development of σ_{ij} round a given point of the element. It is well known that

$$\left|\sigma_{ij} - \hat{\sigma}_{ij}\right| \le C_{ij} h^r \text{ in e,}$$

and this implies

$$\left|\int_{\partial e} \sigma_{ij} n_j \Delta v_i^h dS\right| \le \left|\int_{\partial e} (\sigma_{ij} - \hat{\sigma}_{ij}) n_j \Delta v_i^h dS\right| + \left|\int_{\partial e} \hat{\sigma}_{ij} n_j \Delta v_i^h dS\right|$$

$$\le h^r \left|\int_{\partial e} C_{ij} n_j \Delta v_i^h dS\right|$$

where a h^r factor effectively appears. This is the fundamental idea of the patch test.

8.7.6.- A more complete proof

The above argument is not complete, as doubts may remain on the actual order of h. In fact, the number of elements is $\mathcal{O}\left(\frac{1}{h^2}\right)$ for plane problems and $\mathcal{O}\left(\frac{1}{h^3}\right)$ in space, so that by summing the element contributions, the $\mathcal{O}(h^r)$ could well be destroyed. Fortunately, it is not the case, at least under certain conditions which are generally verified.

a) Firstly, one may write

$$\left|C_{ij} n_j \Delta v_i^h\right| \le \left(C_{ik} n_k C_{il} n_l\right)^{1/2} \left(\Delta v_i^h \Delta v_i^h\right)^{1/2} \le C_1 \left(\Delta v_i^h \Delta v_i^h\right)^{1/2} \qquad \text{(a)}$$

b) Let us now consider Δv_i^h. It is the difference between v_i^h and its interpolate $\tilde{v}_i^h$ which is of the form

$$\tilde{v}_i^h = \sum_k L_k q_k$$

where L_k are interpolation functions and q_k generalized displacements verifying

$$|q_k| \le C_2 \max_e \left|v_i^h\right|$$

so that

$$\left|\tilde{v}_i^h\right| \le \left|\sum_k L_k\right| C_2 \max_e \left|v_i^h\right| = C_3 \max_e \left|v_i^h\right|$$

and finally,

$$\left|\Delta v_i^h\right| \le \left|v_i^h\right| + \left|\tilde{v}_i^h\right| \le (1 + C_3) \max_e \left|v_i^h\right| = C_4 \max_e \left|v_i^h\right| \tag{b}$$

that is, Δ is a bounded operator for the L^∞-norm.

c) This, however, is not sufficient. *One has to assume that* $\Delta v_i^h = 0$ *whenever* v_i^h *is a constant* on the element. This is generally the case in practical applications. Then, choosing some point P_o in the element, one has

$$\Delta v_i^h = \Delta\left(v_i^h - v_i^h(P_o)\right)$$

and from (b),

$$\left|\Delta v_i^h\right| \le C_4 \left|v_i^h - v_i^h(P_o)\right|$$

But it is clear that

$$\left|v_i^h - v_i^h(P_o)\right| \le h \max_e \left|grad\ v_i^h\right| \le h \max_e \left(D_j v_i^h D_j v_i^h\right)^{1/2}$$

so that

$$\left|\Delta v_i^h\right| \le C_4 \max_e \left(D_j v_i^h D_j v_i^j\right)^{1/2}$$

and

$$\left(\Delta v_i^h \Delta v_i^h\right)^{1/2} \le C_4 \sqrt{3} \max_e \left(D_j v_i^h D_j v_i^h\right)^{1/2} = C_5 \max_e \left(D_j v_i^h D_j v_i^h\right)^{1/2} \tag{c}$$

d) We now use the fact that the set of displacement variations v_i^h on the element is *a finite dimensional space* (that is v_i^h depends on a finite number of parameters). It is a classical result that on such a space, all norms are equivalent. So,

$$\max_e \left(D_j v_i^h D_j v_i^h\right)^{1/2} \le C_5 \left(\frac{1}{V_e} \int_e D_j v_i^h D_j v_i^h dV\right)^{1/2} = C_5 V_e^{-1/2} \left\|Dv^h\right\|_e \tag{d}$$

where the dimensional factor $V_e^{-1/2}$ is necessary in order to obtain a C_5 constant which does not depend on h.

e) Assembling results (a) to (d) and integrating on the element boundary, one obtains

$$\int_{\partial e} C_{ij} n_j \Delta v_i^h dS \leq C_6 \, V_e^{-1/2} \, \|Dv\|^2 \, . \, S_e \, h$$

where S_e is the measure of the boundary. Now, for reasonably shaped elements,

$$S_e h \leq C_7 V_e$$

so that

$$h^r \int_{\partial e} C_{ij} n_j \Delta v_i^h dS \leq C_8 h^r V_e^{1/2} \, \|Dv^h\|_e$$

Finally, summing on the elements leads to

$$\mathcal{T} = C_8 h^r \sum_e \left(V_e^{1/2} \|Dv^h\|_e \right) \leq C_8 h^r \left(\sum_e V_e \right)^{1/2} \left(\sum_e \|Dv^h\|_e^2 \right)^{1/2} = C_8 h^r V^{1/2} \|Dv^h\|_V ,$$

which is precisely the requirement (8.10) for a h^r – convergence of stresses.

8.7.7.- The case of fourth order problems

The case of fourth order problems, which arises with Kirchhoff plates, may be treated in a similar way. The only difference is that Δw^h has to be zero for any linear w^h, which is also the most general case.

8.7.8.- Applications

Practically, two cases are possible.

a) The patch test is verified at the element level, due to a compensation between different interfaces (*element patch test*). We will see that it is so with the elements of Adini and Wilson. In this case, the patch test depends on a particular shape of the element (here rectangular) and is not passed for other shapes.

b) The patch test is verified interface by interface (*interface patch test*). In this case, the shape of the element is arbitrary.

[A third case also exists, where the patch test has to be made on a patch of elements. This rare case has no really useful application, because it leads to severe restrictions on the obtainable structural shapes (Zienkiewicz element, [17])].

Note finally that the due work compensation on S_2, leading to (8.14), necessitates that surface tractions $\bar{t}_i$ are applied *on the conforming part of the displacement*, as prescribed in section 8.6.

8.8. Patch test on ADINI's element

The first step in verifying a patch test is to isolate the nonconforming part of the displacement. In Adini's element, the incompatibility is only relative to the slopes. On a vertical side, the slope is by (8.2)

$$\frac{\partial w}{\partial x} = (\alpha_2 + 2\alpha_4 x + 3\alpha_7 x^2) + (\alpha_5 + 2\alpha_8 x + 3\alpha_{11} x^2)y + \alpha_9 y^2 + \alpha_{12} y^3$$

It is connected at both ends, it is, at y = 0 and y = b. These connections imply the continuity of the linear interpolation of $\frac{\partial \tilde{w}}{\partial x}$, which is

$$\frac{\partial \tilde{w}}{\partial x} = (\alpha_2 + 2\alpha_4 x + 3\alpha_7 x^2) + (\alpha_5 + 2\alpha_8 x + 3\alpha_{11} x^2)y + \alpha_9 by + \alpha_{12} b^2 y \qquad (8.16)$$

Consequently, the nonconformint part of the slope is

$$\Delta\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} - \frac{\partial \tilde{w}}{\partial x} = \alpha_9 (y^2 - by) + \alpha_{12}(y^3 - b^2 y) \qquad (8.17)$$

Similarly, from (8.1),

$$\frac{\partial w}{\partial y} = \alpha_3 + \alpha_5 x + 2\alpha_6 y + \alpha_8 x^2 + 2\alpha_9 xy + 3\alpha_{10} y^2 + \alpha_{11} x^3 + 3\alpha_{12} xy^2$$

and, on a horizontal side, this may be written

$$\frac{\partial w}{\partial y} = (\alpha_3 + 2\alpha_6 y + 3\alpha_{10} y^2) + (\alpha_5 + 2\alpha_9 y + 3\alpha_{12} y^2)x + \alpha_8 x^2 + \alpha_{11} x^3 \qquad (8.18)$$

If the side is of length a, the linear interpolation, which is connected, is given by

$$\frac{\partial \tilde{w}}{\partial y} = (\alpha_3 + 2\alpha_6 y + 3\alpha_{10} y^2) + (\alpha_5 + 2\alpha_9 y + 3\alpha_{12} y^2)x + \alpha_8 ax + \alpha_{11} a^2 x \qquad (8.19)$$

and the nonconforming part of the slope is

$$\Delta\frac{\partial w}{\partial y}=\frac{\partial w}{\partial y}-\frac{\partial \tilde{w}}{\partial y}=\alpha_8(x^2-ax)+\alpha_{11}(x^3-a^2x) \tag{8.20}$$

Considering a constant moment field (M_{xx}, M_{yy}, M_{xy}), let us compute the incompatibility work, with the aid of fig. 52.

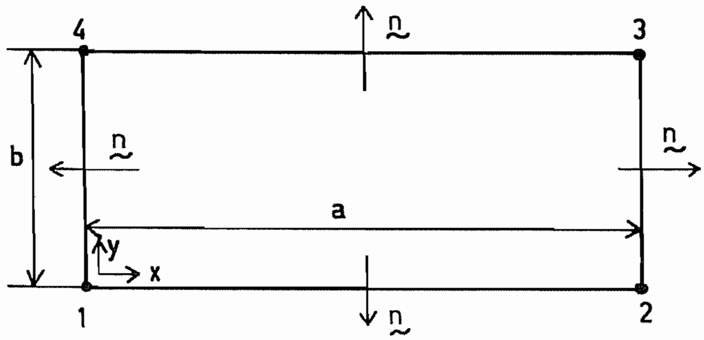


Fig. 52

a) Interface 1-2

n = (0,-1)

$$M_n=M_{xx}n_x^2+2M_{xy}n_xn_y+M_{yy}n_y^2=M_{yy}$$
$$\Delta\frac{\partial w}{\partial n}=-\Delta\frac{\partial w}{\partial y}$$
ds = dx

$$\mathcal{T}_{12}=-\int_o^a M_{yy}\left[\alpha_8(x^2-ax)+\alpha_{11}(x^3-a^2x)\right]dx \tag{8.21}$$

b) Interface 2-3

n = (1,0)

$$M_n=M_{xx}$$
$$\Delta\frac{\partial w}{\partial n}=\Delta\frac{\partial w}{\partial x}$$
ds = dy

$$\mathcal{T}_{23}=\int_o^b M_{xx}\left[\alpha_9(y^2-by)+\alpha_{12}(y^3-b^2y)\right]dy \tag{8.22}$$

c) Interface 3-4

n = (0,1)

$$M_n=M_{xx}$$

$$\Delta\frac{\partial w}{\partial n}=\Delta\frac{\partial w}{\partial y}$$

ds = - dx

$$\mathcal{T}_{34}=-\int_a^o M_{yy}\left[\alpha_8(x^2-ax)+\alpha_{11}(x^3-a^2x)\right](-dx)=\int_o^a M_{yy}\left[\alpha_8(x^2-ax)+\alpha_{11}(x^3-a^2x)\right]dx$$
$$=-\mathcal{T}_{12} \qquad (8.23)$$

d) Interface 4-1

n = (-1,0)

$$M_n = M_{xx}$$
$$\Delta\frac{\partial w}{\partial n}-\Delta\frac{\partial w}{\partial x}$$

ds = - dy

$$\mathcal{T}_{41}=-\int_b^o M_{xx}\left[\alpha_{98}(y^2-by)+\alpha_{12}(y^3-b^2y)\right](-dy)=-\mathcal{T}_{23} \qquad (8.24)$$

Consequently,

$$\mathcal{T}=\mathcal{T}_{12}+\mathcal{T}_{23}+\mathcal{T}_{34}+\mathcal{T}_{41}=0 \qquad (8.25)$$

and the patch test is passed. Analyzing this result, one finds that it is due to the fact that the incompatibility $\Delta\frac{\partial w}{\partial y}$ does not depend on x and symmetrically, $\Delta\frac{\partial w}{\partial x}$ does not depend on y.

8.9. Pach test on Wilson's element

In this case, the nonconforming parts of the displacements are

- on a horizontal side :

$$\Delta u = \beta_1\, x(x - a) \qquad \Delta v = \beta_3\, x(x - a) \qquad (8.26)$$

- on a vertical side :

$$\Delta u = \beta_2\, y(y - b) \qquad \Delta v = \beta_4\, y(y - b) \qquad (8.27)$$

Considering a constant stress field (σ_x, σ_y, τ_{xy}), the incompatibility work may be computed as (fig. 52)

a) Interface 1-2

$\mathbf{n} = (0,-1)$ $\qquad$ $t_x = n_x\sigma_x + n_y\tau_{xy} = -\tau_{xy}$
$t_y = n_x\tau_{xy} + n_y\sigma_y = -\sigma_y$
$ds = dx$

$$\mathcal{T}_{12} = -\int_o^a \left[\tau_{xy}\beta_1 + \sigma_y\beta_3\right] x(x-a)\, dx \tag{8.28}$$

b) Interface 2-3

$\mathbf{n} = (1,0)$ $\qquad$ $t_x = \sigma_x$
$t_y = \tau_{xy}$
$ds = dy$

$$\mathcal{T}_{23} = -\int_o^b \left[\sigma_x\beta_2 + \tau_{xy}\beta_4\right] y(y-b)\, dy \tag{8.29}$$

c) Interface 3-4

$\mathbf{n} = (-1,0)$ $\qquad$ $t_x = \tau_{xy}$
$t_y = \sigma_y$
$ds = -dx$

$$\mathcal{T}_{34} = -\int_a^o \left[\tau_{xy}\beta_1 + \sigma_y\beta_3\right] x(x-a)\,(-dx) = -\mathcal{T}_{12} \tag{8.30}$$

d) Interface 4-1

$\mathbf{n} = (-1,0)$ $\qquad$ $t_x = -\sigma_x$
$t_y = -\tau_{xy}$
$ds = -dy$

$$\mathcal{T}_{41} = -\int_b^o \left[\sigma_x\beta_2 + \tau_{xy}\beta_4\right] x(x-a)\,(-dy) = -\mathcal{T}_{23} \tag{8.31}$$

Summing these four contributions,

$$\mathcal{T} = \mathcal{T}_{12} + \mathcal{T}_{23} + \mathcal{T}_{34} + \mathcal{T}_{41} = 0 \tag{8.32}$$

and *the patch test is passed.* Here also, the success is due to a compensation of the contributions of opposite sides.

8.10. Patch test on Morley's element

Here, an interface patch test may be done. Considering a constant moment field (M_x, M_y, M_{xy}), the corresponding boundary loads are

- the corner loads $Z_i = \Delta(M_{nt})$
- a constant M_n on each side
- a zero $K_n = T_n + \dfrac{\partial M_{nt}}{\partial t}$

The incompatibility work on an interface 1-2, say, is

$$\mathcal{T} = Z_1 \Delta w_1 + Z_2 \Delta w_2 + M_n \int_{(1)}^{(2)} \Delta \frac{\partial w}{\partial n} ds \tag{8.33}$$

But precisely, $\Delta w_1 = \Delta w_2 = 0$ as the displacement is connected at the nodes. Furthermore, as w is of degree 2, $\dfrac{\partial w}{\partial n}$ is of degree 1, so that at the midpoint of the interface, the connected value is

$$\left(\frac{\partial w}{\partial n}\right)_{midpoint} = \frac{1}{l_{12}} \int_{(1)}^{(2)} \frac{\partial w}{\partial n} ds \quad .$$

from which follows

$$\int_{(1)}^{(2)} \Delta \frac{\partial w}{\partial n} ds = 0$$

and *an interface patch test is passed.* Comparing to the two preceeding examples, one can see that here, the work is zero for each interface, not from a compensation between different interfaces.

8.11. Systematic development of nonconforming plate elements which pass the patch test

Systematizing interface patch-tests, FRAEIJS de VEUBEKE developed a very effective family of nonconforming plate elements [33].

8.11.1.- Elements passing a patch test of order 0

Considering a constant moment field, the side forces are

- corner forces Z;
- constant normal moment M_n

The work of incompatibilities of an interface 1-2 of length l is then

$$Z_1 \Delta w_1 + Z_2 \Delta w_2 + M_n \int_o^l \Delta \frac{\partial w}{\partial n} ds \tag{8.34}$$

It vanishes if and only if

$\Delta w_1 = 0$, $\Delta w_2 = 0$

$$\int_o^l \Delta \frac{\partial w}{\partial n} ds = 0$$

and these conditions are fulfilled if the following connectors are used

- nodal displacements
- mean interface slopes $\frac{1}{l}\int_o^l \frac{\partial w}{\partial n} ds$

These connections may be defined *whatever the degree of the interior field*, loeading to a O(h) convergence. The triangular element of this family, FDVTO, is represented on fig. 53 (→ are mean values of the slopes). It requires at least 6 internal d.o.f., it is, a second order field. It is equivalent to Morley's element. More interesting is the quadrilateral version FDVQO (fig. 53) which, having 8 connectors, requires a 3d-degree displacement. There are in this case two bubble modes, whose definition is somewhat difficult. This question will be considered later.

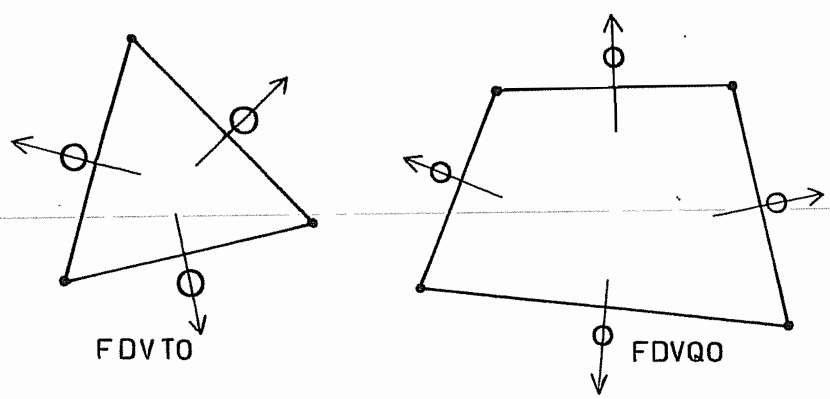


Fig. 53

8.11.2.- Elements passing a patch-test of order 1

The side forces corresponding to a first order moment field are

- corner forces Z_i
- linear normal moment $M_n = \left(1-\frac{s}{l}\right)M_{n1} + \frac{s}{l}M_{n2}$
- constant Kirchhoff's load K_n

On an interface 1-2, the work of incompatibillities is then

$$Z_1\Delta w_1 + Z_2\Delta w_2 + K_n\int_o^l \Delta w ds + M_{n1}\int_o^l\left(1-\frac{s}{l}\right)\Delta\frac{\partial w}{\partial n}ds + M_{n2}\int_o^l \frac{s}{l}\Delta\frac{\partial w}{\partial n}ds \qquad (8.35)$$

Its vanishing supposes

$\Delta w_1 = 0 \rightarrow$ connect w_1
$\Delta w_2 = 0 \rightarrow$ connect w_2

$$\int_o^l \Delta w\, ds = 0 \rightarrow connect\ \frac{1}{l}\int_o^l w\, ds$$

$$\int_o^l \left(1-\frac{s}{l}\right)\Delta\frac{\partial w}{\partial n}ds = 0 \rightarrow connect\frac{1}{l}\int_o^l\left(1-\frac{s}{l}\right)\frac{\partial w}{\partial n}ds$$

$$\int_o^l \frac{s}{l}\Delta\frac{\partial w}{\partial n}ds = 0 \rightarrow connect\frac{1}{l}\int_o^l \frac{s}{l}\frac{\partial w}{\partial n}ds$$

Elements of this family exhibit a $O(h^2)$ convergence, comparable with CQ and HCT elements. The triangular version (fig. 54) makes use of 12 connectors. A fourth-degree displacement field (15 parameters) is thus necessary, and implies 3 bubble modes. The quadrilateral version involves 16 connectors, so that a 5th-degree displacement field (21 parameters) is necessary. There are then 5 bubble modes.

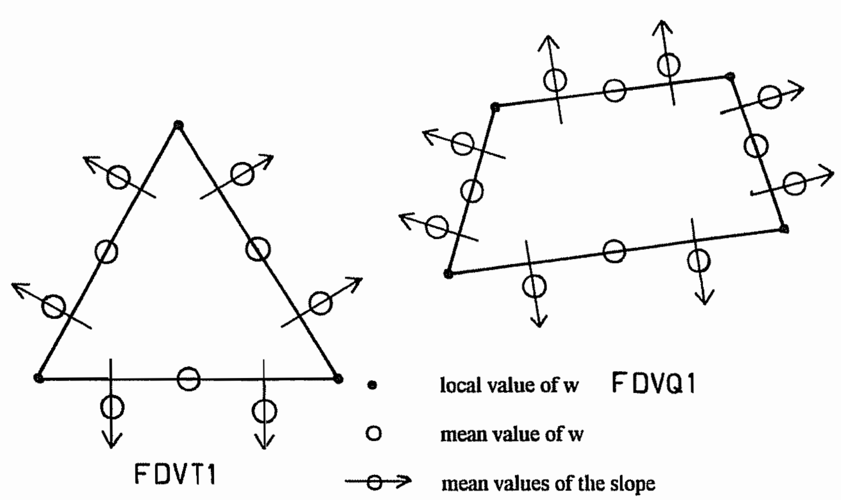


Fig. 54

8.12. Plane shell elements

Plane shell elements are obtained by combining a membrane element to a plate element. In order to obtain 3-dimensional connections, it is first necessary to convert the plate slopes in rotations, by the following rule (fig. 55)

$$\varphi_x = \frac{\partial w}{\partial y} \qquad \varphi_y = -\frac{\partial w}{\partial x} \tag{8.36}$$

Unfortunately, conforming plates and conforming membranes are not suitably connectable at a right angle. Fig. 56 illustrates the possible opening of such a connection which is by no means compatible. The reason is visibly the use of node rotations in the plate elements. This very discouraging result was the state-of-the-art before the development of the nonconforming elements described in the preceeding section (if one excepts one mixed element due to IDELSOHN, [39]). Precisely, these elements have displacement connections that are not very different from those of membrane elements. This was the starting point of the development of nonconforming membrane elements by SANDER and BECKERS [34].

8.13. Non conforming membrane elements

Here, the simplest interface patch-test, corresponding to constant element stresses, is on an interface 1-2

$$t_i \int_o^l \Delta u_i \; dS = 0 \tag{8.37}$$

with

$t_i = n_j \sigma_{ji}$

It is trivially verified if the mean displacement value

$$\tilde{u}_i = \frac{1}{l} \int_o^l u_i \; ds \tag{8.38}$$

is connected. However, as illustrated on fig. 57, such a connection is insufficient, as the possibility of scissor-like displacements remains. Ellipticity, it is the fact that the energy is positive except for rigid body motions, is thus not guaranteed.

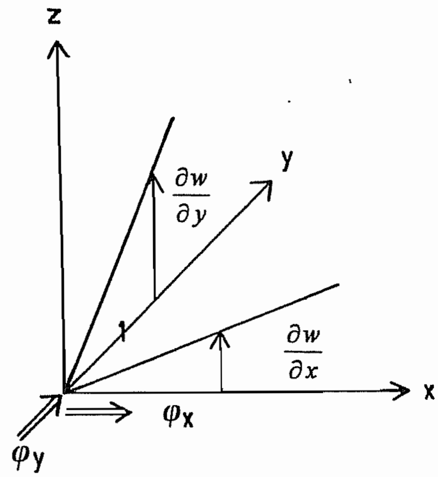


Fig. 55 : From slopes to rotations

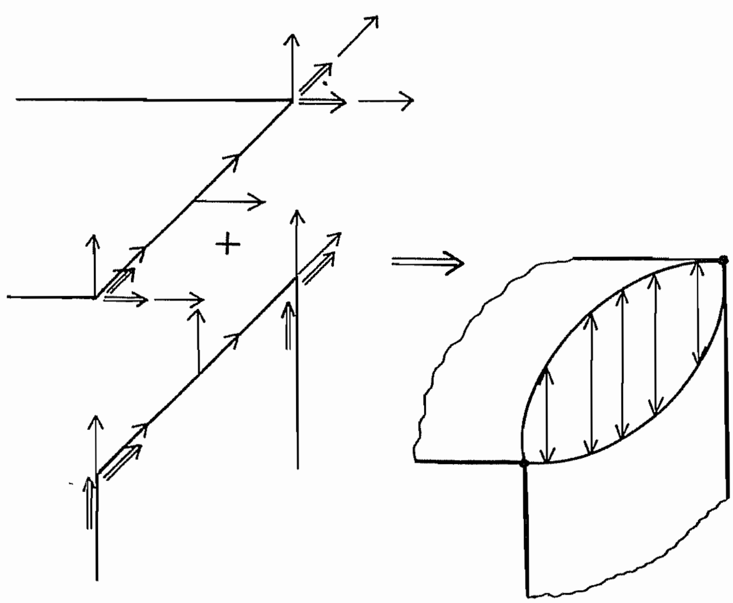


Fig. 56 : Possible opening between plane shell elements

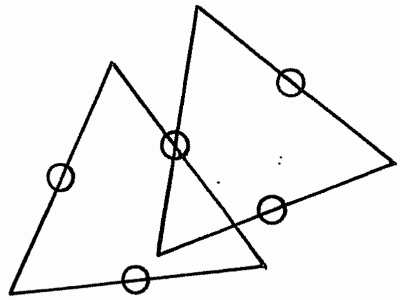


Fig. 57

The easiest way to ensure ellipticity is to connect the *nodes*. So, *if node connections are combined with mean interface values, an element will be obtained which simultaneously guarantees ellipticity and passes the patch test*. Comparing with finite difference shemes, the patch test may be viewed as a *consistency* condition and the ellipticity is to be interpreted as a *stability* condition.

The triangular element SBTO obtained by this way is represented on fig. 58. Its 2 x 6 connectors require displacement fields of degree 2 at least. At the second degree, this element is conforming and exhibits a $O(h^2)$ convergence. At higher degrees, it is nonconforming and its order of convergence reduces to O(h), from it passes a patch test of order 0.

The most interesting fact is that a quadrilateral element can be developped by the same technique. As illustrated in fig. 59, its 2 x 8 connectors implie the use of a 3d-degree displacement field (2 x 10 parameters). There are thus 2 x 2 bubble modes. The convergence is O(h). This element will be called SBQO.

Elements passing higher order patch testes could be envisaged, but were never coded.

8.14. How the use of nonconforming plate and membrane elements solves the problem of angular connections of plane shells

Figure 60 illustrates the fact that FDVQ1 plates and SBQO membranes, as assembled, lead to consistent connections of plane shells. This is perhaps the most spectacular result of the theory of nonconforming elements.

8.15. Morley's plate as a stabilizator of membrane idealizations of 3-dimensional shells

For cost reasons, it is a common practice to idealize a shell with membrane elements. However, the fact that membranes have no normal stiffness leads to a lot of kinematical or quasi-kinematical modes. This can be remedied by adding to the membrane a Morley or a FDVQO element, which only implies one supplementary rotation on each interface and guarantees ellipticity.

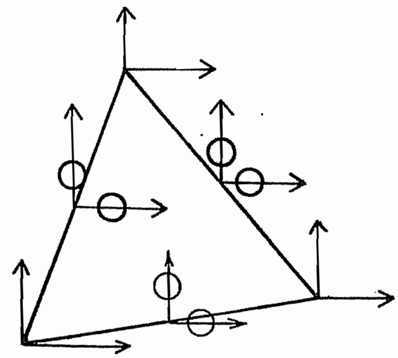


Fig. 58 : Element SBTO

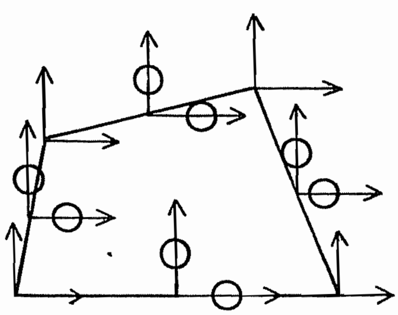


Fig. 59 : Element SBQO

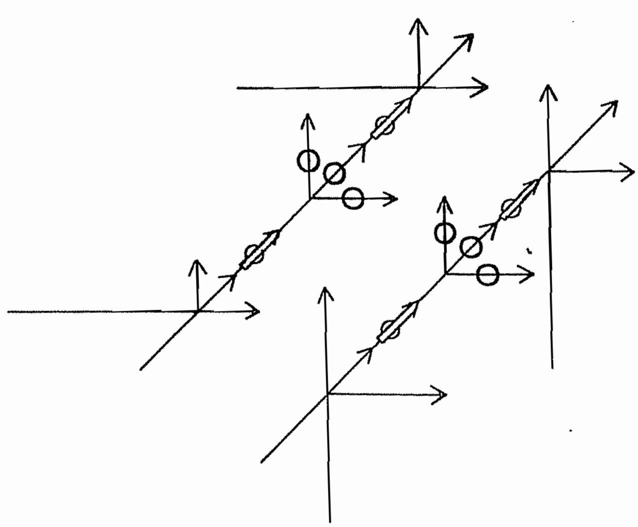


Fig. 60

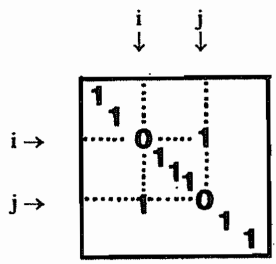
8.16. Automatic bubble detection

As emphasized in the preceeding sections, nonconforming elements involve a lot of bubble modes whose shape may be very difficult to imagine. It is therefore useful to develop an automatic way of treating bubble modes.

8.16.1.- Permutation matrix

Let us consider the square matrix

P(i,j)=



(8.39)

where all undefined elements are zero's. If A is a given matrix of columns c_k,

$$A = [c_1, c_2, \ldots, c_n]$$

One has

$$AP(i,j) = [c_1, \ldots, \underset{\substack{\uparrow \\ i}}{c_j}, \ldots, \underset{\substack{\uparrow \\ j}}{c_i}, \ldots c_n] \tag{8.40}$$

it is to say, columns i and j are permutated. One says that P(i,j) is a permutation matrix. The fundamental properties of P(i,j) are

$$\begin{cases} P(i,j)^T = P(i,j) \\ P(i,j)\; P^T(i,j) = I \end{cases} \tag{8.41}$$

Furthermore, a *pre-multiplication* of P(i,j) on a matrix A leads to the permutation of lines i and j of A.

Let us now consider the matrix

$$P = P(1, i_1)\, P(2, i_2) \ldots P(n-1, i_{n-1}) \tag{8.42}$$

AP is thus the matrix obtained by successively permutating columns 1 and i_1, 2 and i_2, ..., (n-1) and i_{n-1}. It is easy to verify that

$$P P^T = I. \quad , \quad P^T P = I$$

Considering any invertible matrix, one has

$$A^{-1} = (APP^T)^{-1} = (APP^{-1})^{-1} = P(AP)^{-1}$$
$$. = P(1, i_1) \dots P(n-1, i_{n-1}) (AP)^{-1}$$

So, if a matrix A has been inverted after column permutations, the lines of the obtained inverse $(AP)^{-1}$ have to be permutated *in the inverse order*, it is, [(n-1), i_{n-1}] at first and [1, i_1] at last.

8.16.2.- Connection with bubble modes

In the case of non conforming elements, bubble modes have no simple analytic expression and it is not easy to find internal displacements that render the connection square and invertible in every case.

Another way consists to solve the equation

$$C\, a = q \qquad (8.43)$$

with C rectangular, with n_q lines and n_a columns. C is supposed to be of maximum rank n. It is therefore possible to find a system of column permutations P so that

$$\hat{C} = C\, P = \left[\hat{C}_I \ \hat{C}_{II}\right] \qquad (8.44)$$

with $\hat{C}_I$ square-invertible. System (8.43) is then equivalent to

$$C\, P\, P^T\, a = q$$

or

$$\hat{C}\, \hat{a} = q \qquad (8.45)$$

with

$$\hat{a} = P^T a \qquad (8.46)$$

System (8.45) may be set in the form

$$\hat{C}_I \hat{a}_I + \hat{C}_{II} \hat{a}_{II} = q$$

from which

$$\hat{a}_I = \hat{C}_I^{-1} q - \hat{C}_I^{-1} \hat{C}_{II} \hat{a}_{II} \tag{8.47}$$

In this relation, $\hat{a}_{II}$ may take any arbitrary values, independently of the connected values q. $\hat{a}_{II}$ is thus a suitable definition of the bubble modes, and one may set

$$q_b = \hat{a}_{II} \tag{8.48}$$

This leads to the system

$$\begin{cases} \hat{a}_i = \hat{C}_I^{-1} q - \hat{C}_I^{-1} \hat{C}_{II} q_b \\ \hat{a}_{II} = q_b \end{cases}$$

or, in matrix form,

$$\begin{bmatrix} \hat{a}_I \\ \hat{a}_{II} \end{bmatrix} = \begin{bmatrix} \hat{C}_I^{-1} & -\hat{C}_I^{-1} \hat{C}_{II} \\ 0 & I \end{bmatrix} \begin{bmatrix} q \\ q_b \end{bmatrix} \tag{8.49}$$

The general solution of (8.43) is thus

$$a = P\hat{a} = P \begin{bmatrix} \hat{C}_I^{-1} & -\hat{C}_I^{-1} \hat{C}_{II} \\ 0 & I \end{bmatrix} \begin{bmatrix} q \\ q_b \end{bmatrix} \tag{8.50}$$

and the connection problem is solved, with an automatic definition of the bubbles.

8.16.3.- Relation with Gauss inversion

Gauss inversion may be presented in the following manner. Let

$$A = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

be a linear system, and find the successive matrices A(1), A(2)... such that

$$A(1) \begin{bmatrix} y_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A(2) \begin{bmatrix} y_1 \\ y_2 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}, \ldots$$

It is clear that

$$A(n) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

that is to say that A(n) = A^{-1}. Supposing that we are at a certain stage of this formation, we have thus transformed

$$\begin{bmatrix} A_{PP} & A_{PQ} \\ A_{QP} & A_{QQ} \end{bmatrix} \begin{bmatrix} x_P \\ x_Q \end{bmatrix} = \begin{bmatrix} y_P \\ y_Q \end{bmatrix}$$

in a system where y_P and x_P are permutated. This system is easy to obtain. From

$$A_{PP}x_P + A_{PQ}x_Q = Y_P$$

one immediately obtains

$$x_P = A_{PP}^{-1} y_P - A_{PP}^{-1} A_{PQ} x_Q$$

The second equation reduces then to

$$A_{QP} A_{PP}^{-1} y_P - A_{QP} A_{PP}^{-1} A_{PQ} x_Q + A_{QQ} x_Q = y_Q$$

so that the equivalent system is

$$\begin{bmatrix} A_{PP}^{-1} & -A_{PP}^{-1} A_{PQ} \\ A_{QP} A_{PP}^{-1} & A_{QQ} - A_{QP} A_{PP}^{-1} A_{PQ} \end{bmatrix} \begin{bmatrix} y_P \\ x_Q \end{bmatrix} = \begin{bmatrix} x_P \\ y_Q \end{bmatrix} \tag{8.51}$$

Let us apply this result to the permutated connection matrix completed with (n_a - n_q) lines of zeros, it is

$$\begin{bmatrix} \hat{C}_I & \hat{C}_{II} \\ 0 & 0 \end{bmatrix} \tag{8.52}$$

After n_q pivoting operations, one obtains from (8.50) the matrix

$$\begin{bmatrix} \hat{C}_I^{-1} & -\hat{C}_I^{-1} \hat{C}_{II} \\ 0 & 0 \end{bmatrix} \tag{8.53}$$

to which we have to add a unit matrix at the right bottom side to obtain (8.49). The Gauss algorithm is then stopped, and the inverse permutations are made classically.

This algorithm, developed by the author, is adopted in the SAMCEF code [40] for nonconforming elements.

Exercise

Suppose that a hexagonal membrane element has to be developed. Show that this is possible in the frame of nonconforming elements passing the patch test. What is the necessary degree of the displacements ? How many bubble modes will be encountered ? What is the order of convergence ?

CHAPTER 9

DUAL ANALYSIS

9.1. Introduction

After computing an approximate solution by finite elements, the question arises of what is the accuracy of the computed solution. Among the numerous responses which were temptated to this important question, a special mention is due to *dual analysis*. This method, introduced by FRAEIJS de VEUBEKE [41], and applied by himself and his co-workers [42 to 46], consists to compare two analyses of the same problem, the first one, of the displacement type and the second one, of the equilibrium type, and to deduce from the respective energies an useful error bound. FRAEIJS de VEUBEKE's analysis was however restricted to particular boundary conditions, i.e. zero prescribed displacements or zero applied loads. These restrictions were due to a point of view where energy bounds played a central role. More recently, DEBONGNIE, ZHONG and BECKERS [47, 48] reformulated this question in a more general way, where energy bounds do not play the central role. The result is a dual error bound which is valid whatever be the boundary conditions. The former results of Fraeijs de Veubeke are then particular cases of the general theory. The present chapter follows this way which, incidentally, is probably the simplest one.

9.2. The displacement approach

Admissible displacement fields are those displacement fields that satisfy a priori the boundary condition $u_i = \bar{u}_i$ on S_1. An *admissible displacement variation* δu is then defined as the difference of two admissible fields. Consequently, one must have

$$\delta u_i = 0 \text{ on } S_1 \tag{9.1}$$

The displacement approach consists in finding, among all admissible displacement fields, the particular field u that minimizes the total potential energy

$$\mathscr{E}(u) = \mathscr{U}(u) + \mathscr{P}(u) \tag{9.2}$$

where

$$\mathscr{U}(u) = \frac{1}{2}\int_V C_{ijkl}\, \varepsilon_{ij}(u)\, \varepsilon_{kl}(u)\, dV \tag{9.3}$$

is the *strain energy*, and

$$\mathscr{P}(u) = -\int_V f_i u_i dV - \int_{S_2} \bar{t}_i u_i dS \tag{9.4}$$

is the *potential of prescribed loads*. Varying functional (9.2) with respect to u leads to equilibrium equations. In other words, the exact equilibrated solution u of the elastic problem is the only one verifying the condition

$$\delta\, \mathscr{E}(u;\delta u) = 0 \tag{9.5}$$

for any admissible displacement variation δu.

Let us now denote U the set of admissible displacements and δU the space of admissible variations. Rayleigh-Ritz approximations and in particular, conforming finite element methods, consist to select some subset U_h of U containing displacements u_h and consequently, a subspace δU_h of displacement variations. In a strict displacement model, the kinematical conditions

$$u_{hi} = \overline{u}_i \tag{9.6}$$

have to be verified *exactly*. In other terms, u_h has to be *strictly* admissible. A Rayleigh-Ritz solution is defined by the condition that

$$\delta\, \mathscr{E}(u_h;\, \delta u_h) = 0 \tag{9.7}$$

for any $\delta u_h \in \delta U_h$. But for the following developments, it is not necessary that u_h should be a Rayleigh-Ritz approximation.

Let thus u_h be any approximate displacement verifying (9.6), and let us define

$$\Delta u = u_h - u \tag{9.8}$$

where u is the exact solution. The total potential energy admits the following development

$$\mathscr{E}(u_h) = \mathscr{E}(u + \Delta u) = \mathscr{E}(u) + \delta\mathscr{E}(u; \Delta u) + \frac{1}{2}\,\delta^2\mathscr{E}(\Delta u) \tag{9.9}$$

where

$$\delta^2\mathscr{E}(\Delta u) = \int_V C_{ijkl}\,\varepsilon_{ij}(\Delta u)\varepsilon_{kl}(\Delta u)\; dV \tag{9.10}$$

in the second variation of $\mathscr{E}$. But, from (9.8), Δu is an admissible displacement variation, so that equation (9.5) is true, it is

$$\delta\mathscr{E}(u;\, \Delta u) = 0 \tag{9.11}$$

Therefore,

$$\mathscr{E}(u_h) = \mathscr{E}(u) + \frac{1}{2}\,\delta^2\mathscr{E}(\Delta u) \tag{9.12}$$

$\delta^2\mathscr{E}(\Delta u)$ is an energetic measure of the approximation error, namely, twice the energy of the variation Δu. This fact will be reflected by adopting the norm notation $\|\Delta u\|^2$. The result is thus

$$\|\Delta u\|^2 = 2\left[\mathscr{E}(u_h) - \mathscr{E}(u)\right] \tag{9.13}$$

9.3. The equilibrium approach

Statically admissible stress fields are those stress fields σ that satisfy the equilibrium equations

$$\begin{aligned} &D_j\sigma_{ji} + f_i = 0 && \text{in V} \\ &n_j\sigma_{ji} = \bar{t}_i && \text{on } S_2 \end{aligned} \tag{9.14}$$

On eventual discontinuity interfaces, if index + denotes one side and index - the other one, the condition is

$$(n_j\sigma_{ji})_+ + (n_j\sigma_{ji})_- = 0 \tag{9.15}$$

A *statically admissible stress variation* δσ is defined as the difference between two statically admissible stress fields. This implies

$$\begin{aligned} &D_j\, \delta\sigma_{ji} = 0 && \text{in V} \\ &n_j\, \delta\sigma_{ji} = 0 && \text{on } S_2 \end{aligned} \tag{9.16}$$

Condition (9.15) is maintained. The set of statically admissible stresses will be noted E, and for the space of statically admissible stress variations, the notation δE will be used.

The equilibrium approach consists in choosing, among all statically admissible stress fields, the particular field σ that minimizes the total complementary energy

$$\mathscr{C}(\sigma) = \mathscr{V}(\sigma) + \mathscr{D}(\sigma) \tag{9.17}$$

where

$$\mathscr{V}(\sigma) = \frac{1}{2}\int_V C^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}\,dV \tag{9.18}$$

is the *complementary strain energy* and

$$\mathscr{D}(\sigma) = -\int_{S_1} n_j\sigma_{ji}\bar{u}_i\,dS \tag{9.19}$$

is the *potential of prescribed displacements*; It is well known that this principle leads to compatibility conditions. The exact solution σ thus verifies the condition

$$\delta\mathscr{C}(\sigma;\delta\sigma) = 0 \tag{9.20}$$

for any statically admissible stress variation.

Rayleigh-Ritz approximations consist to adopt a restriction E_h of E, from which naturally derives a subspace δE_h of δE, and to choose $\sigma_h \in E_h$ such that

$$\delta\mathscr{C}(\sigma_h; \delta\sigma_h) = 0 \tag{9.21}$$

whatever be $\delta\sigma_h \in \delta E_h$. In a *strict* equilibrium model, equilibrium equations (9.14) have to be verified *exactly* by the approximate field σ_h. In the same manner as in displacement models, this is the only property that we will require, and it is by no means necessary that σ_h will be a Rayleigh-Ritz approximation

Setting

$$\Delta\sigma_{ij} = \sigma_{hij} - \sigma_{ij} \tag{9.22}$$

leads to the following development of $\mathscr{C}(\sigma_h)$

$$C(\sigma_h) = C(\sigma + \Delta\sigma) = C(\sigma) + \delta C(\sigma; \Delta\sigma) + \frac{1}{2}\delta^2 C(\Delta\sigma) \tag{9.23}$$

with

$$\delta^2 C(\Delta\sigma) = \int_V C^{-1}_{ijkl}\Delta\sigma_{ij}\Delta\sigma_{kl}\,dV \tag{9.24}$$

As both σ and σ_h satisfy the equilibrium equations, their difference $\Delta\sigma$ is a statically admissible stress variation, so that the first variation vanishes and

$$C(\sigma_h) = C(\sigma) + \frac{1}{2}\delta^2 C(\Delta\sigma)$$

$\delta^2\mathscr{C}(\Delta\sigma)$ is an energetic measure of the error, that we may note $\|\Delta\sigma\|^2$, from which results

$$\|\Delta\sigma\|^2 = 2\left[C(\sigma_h) - C(\sigma)\right] \tag{9.25}$$

9.4. The general dual analysis

Displacement and equilibrium approaches lead both to the same solution

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}(u)$$

so that

$$U(u)=\frac{1}{2}\int_V C_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)\,dV=\frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}(u)\,dV$$

$$=\frac{1}{2}\int_V C^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}=\mathcal{V}(\sigma) \qquad (9.26)$$

Furthermore, an integration by parts leads to the relation

$$\int_V \sigma_{ij}\varepsilon_{ij}(u)\,dV=\int_S n_j\sigma_{ji}u_i\,dS-\int_V u_iD_j\sigma_{ji}\,dV$$

and, taking into account the prescribed values of the displacements on S_1 and the equilibrium equations on S_2 and in V, one obtains

$$\int_V \sigma_{ij}\varepsilon_{ij}(u)dV=\int_{S_1} n_j\sigma_{ji}\bar{u}_i dS+\int_{S_2} t_iu_i dS+\int_V f_iu_i dV \qquad (9.27)$$

The first member of this relation may be re-written, from (9.26),

$$\int_V \sigma_{ij}\varepsilon_{ij}(u)dV=\mathcal{U}(u)+\mathcal{V}(\sigma)$$

so that the solution verifies

$$\mathcal{U}(u)+\mathcal{V}(\sigma)=-\mathcal{Q}(\sigma)-\mathcal{P}(u)$$

or, equivalently,

$$\mathcal{E}(u)+\mathcal{C}(\sigma)=0 \qquad (9.28)$$

From this, it is now sufficient to add relations (9.13) and (9.25) to obtain the fundamental result of dual analysis

$$\|\Delta u\|^2+\|\Delta\sigma\|^2=2[\mathcal{E}(u_h)+\mathcal{C}(\sigma_h)] \qquad (9.29)$$

A more refined analysis [47] allows to show that this sum of square errors is also the square of the energetic distance between the two approximations, but this result is of less practical interest.

Practically, it is preferable to work with the square root of (9.29) and to compare it to the energetic norm of the true solution

$$[2\mathcal{U}(u)]^{1/2}=[2\mathcal{V}(\sigma)]^{1/2}\approx[\mathcal{U}(u_h)+\mathcal{V}(\sigma_h)]^{1/2}$$

so as to obtain a relative error measure

$$R.E. = \left(\frac{\|\Delta u\|^2 + \|\Delta \sigma\|^2}{\|u\|^2 + \|\sigma\|^2} \right)^{1/2} \approx \left(\frac{2[\mathscr{E}(u_h) + \mathscr{C}(\sigma_h)]}{2[\mathscr{U}(u_h) + \mathscr{V}(\sigma_h)]} \right)^{1/2} = \left(\frac{\mathscr{E}(u_h) + \mathscr{C}(\sigma)}{\mathscr{U}(u_h) + \mathscr{V}(\sigma_h)} \right)^{1/2} \quad (9.30)$$

It is interesting to note that the evaluation of this relative error only requires very simple computations from the results. One may naturally object that two finite element analyses are necessary to obtain such an error measure. But the present proof never used the assumption that u_h and σ_h should be Rayleigh-Ritz approximations. The only requirement is that u_h and σ_h are admissible. As an example, after a displacement finite element analysis, one may imagine to construct a statically admissible σ_h field, inspired from the displacement analysis, and use the preceeding results. This way was followed by LADEVEZE [49 to 51] and leads to one of the most rigorous methods of a posteriori error evaluation.

9.5. Bounds on the extended total complementary energy [53,54]

9.5.1. General case

Let us now define a new functional, *the extended total complementary energy* $\mathscr{C}^*$, as follows

- For any statically admissible stress field σ_h,

$$\mathscr{C}^*(\sigma_h) = \mathscr{C}(\sigma_h)$$

- For any kinematically admissible displacement field u_h,

$$\mathscr{C}^*(u_h) = -\mathscr{E}(u_h)$$

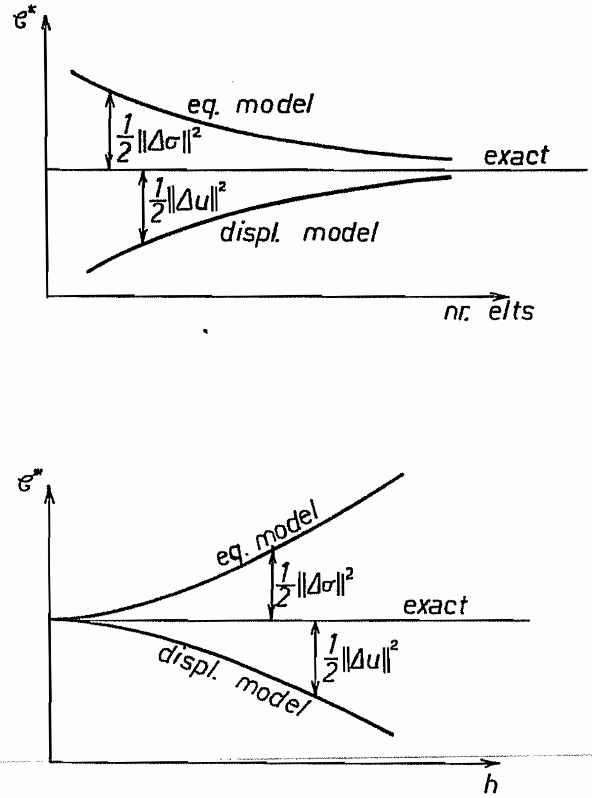
Then, it follows from (9.28) that at the true solution, $\mathscr{C}^*(u) = \mathscr{C}^*(\sigma)$. Moreover, one has

$$\mathscr{C}^*(\sigma_h) \geq \mathscr{C}^*(\sigma), \qquad \mathscr{C}^*(\sigma_h) - \mathscr{C}^*(\sigma) = \frac{1}{2}\|\sigma_h - \sigma\|^2$$

and

$$\mathscr{C}^*(u_h) \leq \mathscr{C}^*(u), \qquad \mathscr{C}^*(u) - \mathscr{C}^*(u_h) = \frac{1}{2}\|u - u_h\|^2$$

This leads to *convergence curves* of more and more refined models, the equilibrium models converging to the exact value of $\mathscr{C}^*$ by decreasing, while the displacement models converge to the exact value of $\mathscr{C}^*$ by increasing. The distance between the two curves is a measure of the convergence (fig. 60bis).



$\mathscr{E}^*$
eq. model
$\frac{1}{2}\|\Delta\sigma\|^2$
exact
$\frac{1}{2}\|\Delta u\|^2$
displ. model
nr. elts
$\mathscr{E}^*$
eq. model
$\frac{1}{2}\|\Delta\sigma\|^2$
exact
$\frac{1}{2}\|\Delta u\|^2$
displ. model
h

Fig. 60bis

9.5.2. Fraeijs de Veubeke's particular cases

The classical dual analysis, as proposed by Fraeijs de Veubeke, was derived from the following supplementary assumptions which guarantee the existence of upper and lower bounds of the energy.

(a) One type of boundary conditions is homogeneous
(b) The approximate homogeneous field is obtained by a Rayleigh-Ritz procedure.

There are thus two cases that have to be considered separately.

(i) Homogeneous prescribed displacements, $\bar{u}_i = 0$

In this case, the solution u is itself an admissible displacement variation, from which follows

$$= \delta\mathscr{E}(u;u) = \int_V C_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)\, dV + \mathscr{P}(u) = 2\mathscr{U}(u) + \mathscr{P}(u) = 0\,,$$

that is,

$$2\mathscr{U}(u) = -\mathscr{P}(u),\ \mathscr{E}(u) = -\mathscr{U}(u)$$

The same is true of the *Rayleigh-Ritz* approximation u_h,

$$2\mathscr{U}(u_h) = -\mathscr{P}(u_h),\ \mathscr{E}(u_h) = -\mathscr{U}(u_h)$$

As a consequence,

$$-\mathscr{U}(u_h) = \mathscr{E}(u_h) \geq \mathscr{E}(u) = -\mathscr{U}(u) \tag{9.31}$$

In the equilibrium model, the potential of prescribed displacement vanishes, so that, for any statically *admissible* stress field σ_h,

$$\mathscr{C}(\sigma) = \mathscr{V}(\sigma) \leq \mathscr{C}(\sigma_h) = \mathscr{V}(\sigma_h) \tag{9.32}$$

Applying the relation (9.28) gives

$$\mathscr{U}(u_h) \leq \mathscr{U}(u) = -\mathscr{E}(u) = \mathscr{C}(\sigma) = \mathscr{V}(\sigma) \leq \mathscr{V}(\sigma_h) \tag{9.33}$$

that is *upper and lower bounds of the energy*, and the error measure becomes

$$\|\Delta u\|^2 + \|\Delta\sigma\|^2 = 2\left[\mathscr{E}(u_h) + \mathscr{C}(\sigma_h)\right] = 2\left[\mathscr{V}(\sigma_h) - \mathscr{U}(u_h)\right] \tag{9.34}$$

In this case, the error is thus measured by the difference between the two obtained values of the elastic energy.

(ii) Homogeneous equilibrium, $f_i = 0$ and $t_i = 0$

In this case, the solution σ is itself a statically admissible stress variation. The same is true of the Rayleigh-Ritz approximation σ_h, so that

$$= \delta\mathscr{C}(\sigma;\sigma) = \int_V C^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}dV + \mathscr{D}(\sigma) = 2\mathscr{V}(\sigma) + \mathscr{D}(\sigma) = 0$$

that is,

$$2\mathscr{V}(\sigma) = -\mathscr{D}(\sigma), \qquad \mathscr{C}(\sigma) = -\mathscr{V}(\sigma)$$

and; similarly,

$$2\mathscr{V}(\sigma_h) = -\mathscr{D}(\sigma_h), \qquad \mathscr{C}(\sigma_h) = -\mathscr{V}(\sigma_h),$$

As a consequence,

$$\mathscr{V}(\sigma_h) \leq \mathscr{V}(\sigma) \tag{9.35}$$

As the potential of the prescribed loads vanishes, any *admissible* displacement field u_h verifies

$$\mathscr{E}(u) = \mathscr{U}(u) \leq \mathscr{E}(u_h) = \mathscr{U}(u_h) \tag{9.36}$$

The *upper and lower bounds of the energy* are now

$$\mathscr{V}(\sigma_h) \leq \mathscr{V}(\sigma) = \mathscr{U}(u) \leq \mathscr{U}(u_h) \tag{9.37}$$

and the cumulative error is given by

$$\|\Delta u\|^2 + \|\Delta\sigma\|^2 = 2[\mathscr{E}(u_h) + \mathscr{C}(\sigma_h)] = 2[\mathscr{U}(u_h) - \mathscr{V}(\sigma_h)] \tag{9.38}$$

The error is also measured by the difference between computed elastic energies, but with the reversed sign.

9.6. The stress function approach

Equilibrium elements were developped by Fraeijs de Veubeke [6] and others, but their description goes beyond the frame of the present course. However, if body forces vanish, an equilibrium approach may be performed with displacement elements, by using the stress function formulation [45].

9.6.1.- Airy function for plane stress

The general solution of the equation

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0$$

is

$$\sigma_x=\frac{\partial A}{\partial y},\ \tau_{xy}=-\frac{\partial A}{\partial x} \tag{9.39}$$

The second equilibrium equation

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

admits the general solution

$$\tau_{xy}=\frac{\partial B}{\partial y},\ \sigma_y=-\frac{\partial B}{\partial x} \tag{9.40}$$

The equality of the two expressions of τ_{xy} necessitates

$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y},$$

an equation whose general solution is of the form

$$A=\frac{\partial \varphi}{\partial y},\ B=-\frac{\partial \varphi}{\partial x} \tag{9.41}$$

From this follows the general form of internally equilibrated stresses, namely,

$$\sigma_x=\frac{\partial^2 \varphi}{\partial y^2},\ \sigma_y=\frac{\partial^2 \varphi}{\partial x\,\partial y},\ \tau_{xy}=-\frac{\partial^2 \varphi}{\partial x\,\partial y} \tag{9.42}$$

The compatibillity equations may be obtained by minimizing the total complementary energy, whose variation is

$$\int_S (\varepsilon_x\delta\sigma_x+\varepsilon_y\delta\sigma_y+\gamma_{xy}\delta\tau_{xy})\,dS-\int_{\mathcal{C}_u}\left[\bar{u}(n_x\delta\sigma_x+n_y\delta\tau_{xy})+\bar{v}(n_x\delta\tau_{xy}+n_y\delta\sigma_y)\right]ds$$

The surface term is

$$\int_S \left(\varepsilon_x \frac{\partial^2 \delta \varphi}{\partial y^2} + \varepsilon_y \frac{\partial^2 \delta \varphi}{\partial x^2} - \gamma_{xy} \frac{\partial^2 \delta \varphi}{\partial x \partial y} \right) dS$$

$$= \text{boundary terms} + \int_S \delta \varphi \left(\frac{\partial^2 \varepsilon_x}{\partial y^2} + \varepsilon_y \frac{\partial^2 \varepsilon_y}{\partial x^2} - \gamma_{xy} \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \right) dS$$

The internal compatibility conditions are thus

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \tag{9.43}$$

Finally, the strain-stress relations are

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{9.44}$$

All these relations are similar to plate equations, where

$$-\chi_x = \frac{\partial^2 w}{\partial x^2} \qquad -\chi_y = \frac{\partial^2 w}{\partial y^2} \qquad -\frac{1}{2}\chi_{xy} = \frac{\partial^2 w}{\partial x \partial y} \tag{9.45}$$

$$M_x = \frac{-Et^3}{12(1-\nu^2)}(\chi_x + \nu \chi_y)$$

$$M_y = \frac{-Et^3}{12(1-\nu^2)}(\chi_y + \nu \chi_x)$$

$$2M_{xy} = \frac{-Et^3}{12(1-\nu^2)} 2(1-\nu) \frac{\chi_{xy}}{2} \tag{9.46}$$

and

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = 0 \tag{9.47}$$

The analogy is given by the following table

Plate	Stress function
w	φ
$-\chi_x$	σ_y
$-\chi_y$	σ_x
$-\frac{1}{2}\chi_{xy}$	$-\tau_{xy}$
M_x	ε_y
M_y	ε_x
$2M_{xy}$	$-\gamma_{xy}$
$\frac{Et^3}{12(1-\nu^2)}$	$\frac{1}{E}$
ν	$-\nu$

To impose boundary tractions, note that

$$T_x = \sigma_x n_x + \tau_{xy} n_y = n_x \frac{\partial^2 \varphi}{\partial y^2} - n_y \frac{\partial^2 \varphi}{\partial x \partial y}$$

and, from the fact that $n_x = t_y$, $n_y = -t_x$, one obtains

$$T_x = t_y \frac{\partial^2 \varphi}{\partial y^2} + t_x \frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial y}\right) \tag{9.48}$$

Similarly,

$$T_y = \tau_{xy} n_x + \sigma_y n_y = -n_x \frac{\partial^2 \varphi}{\partial x \partial y} + n_y \frac{\partial^2 \varphi}{\partial x^2} = -t_x \frac{\partial^2 \varphi}{\partial x^2} - t_y \frac{\partial^2 \varphi}{\partial x \partial y} = -\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right) \tag{9.49}$$

It is thus possible to use plate conforming elements to obtain an equilibrium model in plane stress.

9.6.2.- Southwell functions for a plate

For a plate with a zero pressure load, the equilibrium of shear loads is expressed by the equation

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} = 0$$

whose general solution is

$$T_x = \frac{\partial \Omega}{\partial y}, \quad T_y = -\frac{\partial \Omega}{\partial x} \tag{9.50}$$

The equilibrium equations

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = T_x$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = T_y$$

may thus be re-written in the form

$$\frac{\partial M_x}{\partial x} + \frac{\partial}{\partial y}(M_{xy} - \Omega) = 0$$

$$\frac{\partial}{\partial x}(M_{xy} + \Omega) + \frac{\partial M_y}{\partial y} = 0$$

The general solution of these equations is

$$M_x = \frac{\partial V}{\partial y}, \quad M_{xy} - \Omega = -\frac{\partial V}{\partial x}$$

$$M_{xy} + \Omega = -\frac{\partial U}{\partial y}, \quad M_y = \frac{\partial U}{\partial x} \tag{9.51}$$

or

$$M_x = \frac{\partial V}{\partial y}, M_y = \frac{\partial U}{\partial x}, M_{xy} = -\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)$$

$$\Omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$$

The bending moments are thus similar to strains calculated from a displacement (U, V). The complete setting of the analogy with a displacement formulation of plane stress is leaved to the reader.

9.7. An example of statically admissible analysis

A very instructive example of statically admissible analysis is given by a beam submitted to an uniformly distributed and shear force $T_y = -q$ (fig. 61). Most authors say that this problem is an ill-posed one, because it does not conform to the general reciprocity condition of shear stresses. Such a conclusion cannot be retained as definitive, as the present loading is perfectly regular. The problem is that it implies a *discontinuous solution*. In the frame of the stress function approach, $\tau_{xy} = -\dfrac{\partial^2 \varphi}{\partial x \, \partial y}$ and we are precisely confronted to the singularity theorem (section 7.9) which says that no conforming element can be developped with continuous crossed derivatives. In other words, a solution may be found in the use of a CQ element which precisely exhibits discontinuous crossed derivatives at the nodes.

Traduced in terms of Airy's function, the boundary conditions are given on fig. 62, and made more precise on fig. 63, where use is made of the fact that any substraction of a polynomial of degree 1 to Airy's function has no effect on the stresses. The solution will be constructed as shown in fig. 64. But a fundamental simplification is obtained by noting that boundary conditions are antisymmetric in terms of y. *It is thus reasonnable to think that the same behavior will be found for the function* φ.

Let us begin by element 1. The boundary conditions, at x = l, are

$$\varphi = 0, \quad \frac{\partial \varphi}{\partial x} = qy \tag{9.52}$$

A particular solution to these conditions is obtained by solving the second condition, which leads to

$$\varphi_{part} = q\,xy + C(y)$$

Taking the condition $\varphi = 0$ at x = l in account, one obtains

$$qly + C(y) = 0$$

that is,

$$C(y) = -q\,l\,y.$$

and

$$\varphi_{part} = q(x-l)y = -qhl\frac{y}{h}\left(1-\frac{x}{l}\right) \tag{9.53}$$

To obtain the general solution in element 1 satisfying (9.52), we only have to add a solution of $\varphi = 0$, $\dfrac{\partial \varphi}{\partial x} = 0$ which should be antisymmetrical in terms of y. Such a function is of the form

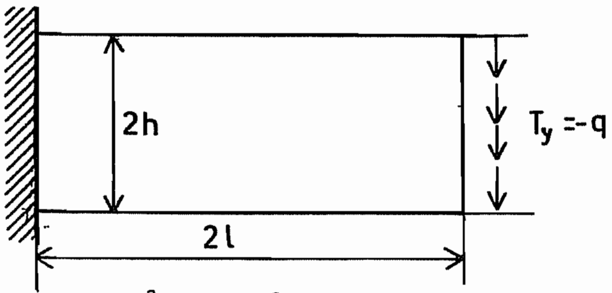


Fig. 61

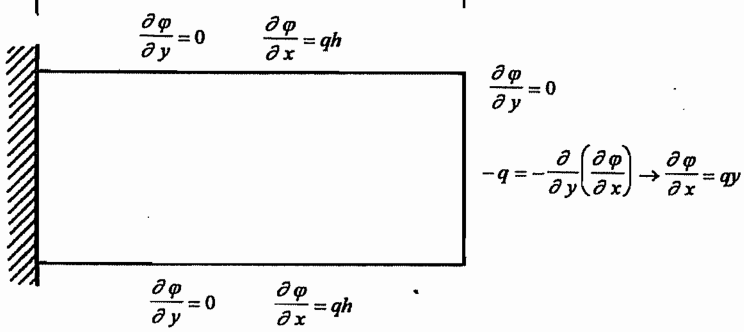


Fig. 62

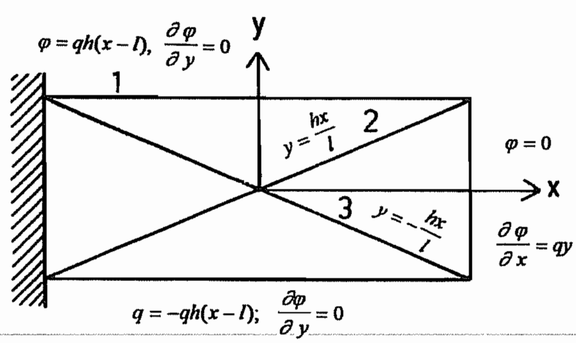


Fig. 63

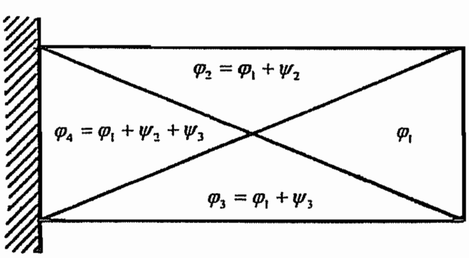


Fig. 64

$$\left(1-\frac{x}{l}\right)^2 \alpha_1 \frac{y}{h},$$

so that

$$\varphi_1 = -qhl\,\frac{y}{h}\left(1-\frac{x}{l}\right)+\alpha_1\left(1-\frac{x}{l}\right)^2\frac{y}{h} \tag{9.54}$$

In element 2, the solution will differ from φ_1 by a multiple of the square of the equation of the line $y=\frac{hx}{l}$, that we also suppose to be a multiple of y. The only form of this difference is

$$\psi_2 = \varphi_2 - \varphi_1 = \alpha_2\,\frac{y}{h}\left(\frac{y}{h}-\frac{x}{l}\right)^2 \tag{9.55}$$

From this,

$$\varphi_2 = -qhl\,\frac{y}{h}\left(1-\frac{x}{l}\right)+\alpha_1\left(1-\frac{x}{l}\right)^2\frac{y}{h}+\alpha_2\,\frac{y}{h}\left(\frac{y}{h}-\frac{x}{l}\right)^2$$

and

$$\frac{\partial\varphi_2}{\partial(y/h)} = -qhl\left(1-\frac{x}{l}\right)+\alpha_1\left(1-\frac{x}{l}\right)^2+\alpha_2\left(\frac{y}{h}-\frac{x}{l}\right)^2+2\alpha_2\,\frac{y}{h}\left(\frac{y}{h}-\frac{x}{l}\right)$$

At the upper boundary, y = h and one must have

$$\varphi_2 = -qhl\left(1-\frac{x}{l}\right)+\alpha_1\left(1-\frac{x}{l}\right)^2+\alpha_2\left(1-\frac{x}{l}\right)^2 = -qhl\left(1-\frac{x}{l}\right)$$

from which $\alpha_1+\alpha_2=0$. Moreover,

$$0=\frac{\partial\varphi_2}{\partial\left(\frac{y}{h}\right)} = -qhl\left(1-\frac{x}{l}\right)+\alpha_1\left(1-\frac{x}{l}\right)^2+\alpha_2\left(1-\frac{x}{l}\right)^2+2\alpha_2\left(1-\frac{x}{l}\right)=0$$

and, as $\alpha_1+\alpha_2=0$,

$$\alpha_2=\frac{qhl}{2},\quad \alpha_1=-\frac{qhl}{2}.$$

One has thus

$$\varphi_1 = -qhl\,\frac{y}{h}\left(1-\frac{x}{l}\right)-\frac{qhl}{2}\frac{y}{h}\left(1-\frac{x}{l}\right)^2 \tag{9.56}$$

and

$$\psi_2 = \frac{qhl}{2}\frac{y}{h}\left(\frac{y}{h}-\frac{x}{l}\right)^2 \tag{9.57}$$

In element 3, the complementary term ψ_3 must be a multiple of the square of the equation of the line 2-3, and we suppose it to be also multiple of y. These considerations give

$$\psi_3 = \varphi_3 - \varphi_1 = \alpha_3\,\frac{y}{h}\left(\frac{y}{h}+\frac{x}{l}\right)^2 \tag{9.58}$$

So,

$$\varphi_3 = -qhl\,\frac{y}{h}\left(1-\frac{x}{l}\right)-\frac{qhl}{2}\frac{y}{h}\left(1-\frac{x}{l}\right)^2+\alpha_3\,\frac{y}{h}\left(\frac{y}{h}+\frac{x}{l}\right)^2$$

and

$$\frac{\partial\varphi_3}{\partial\left(\frac{y}{h}\right)} = -qhl\left(1-\frac{x}{l}\right)-\frac{qhl}{2}\left(1-\frac{x}{l}\right)^2+\alpha_3\left(\frac{y}{h}+\frac{x}{l}\right)^2+2\alpha_3\,\frac{y}{h}\left(\frac{y}{h}+\frac{x}{l}\right)$$

At the lower boundary, y/h = -1 and

$$\varphi_3 = +qhl\left(1-\frac{x}{l}\right)+\frac{qhl}{2}\left(1-\frac{x}{l}\right)^2-\alpha_3\left(1-\frac{x}{l}\right)^2 = qhl\left(1-\frac{x}{l}\right)$$

from which

$$\alpha_3 = +\frac{qhl}{2} \tag{9.59}$$

One has only to verify that

$$\frac{\partial\varphi_3}{\partial\left(\frac{y}{h}\right)} = -qhl\left(1-\frac{x}{l}\right)-\frac{qhl}{2}\left(1-\frac{x}{l}\right)^2+\frac{qhl}{2}\left(1-\frac{x}{l}\right)^2+qhl\left(1-\frac{x}{l}\right)=0,$$

which is the case. The solution is thus

Element 1 : $\varphi_1 = -qhl\,\frac{y}{h}\left(1-\frac{x}{l}\right) - \frac{qhl}{2}\,\frac{y}{h}\left(1-\frac{x}{l}\right)^2$.

Element 2 : $\varphi_2 = \varphi_1 + \psi_2$

$$\psi_2 = \frac{qhl}{2}\,\frac{y}{h}\left(\frac{y}{h}-\frac{x}{l}\right)^2$$

Element 3 : $\varphi_3 = \varphi_1 + \psi_3$

$$\psi_3 = \frac{qhl}{2}\,\frac{y}{h}\left(\frac{y}{h}+\frac{x}{l}\right)^2$$

Element 4 : $\varphi_4 = \varphi_1 + \psi_2 + \psi_3$ (9.60)

Stresses are now easy to compute from

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} = \frac{1}{h^2}\,\frac{\partial^2\varphi}{\partial\left(\frac{y}{h}\right)^2}$$

$$\sigma_y = \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{l^2}\,\frac{\partial^2\varphi}{\partial\left(\frac{x}{h}\right)^2}$$

$$\tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} = -\frac{1}{hl}\,\frac{\partial^2\varphi}{\partial\left(\frac{x}{l}\right)\partial\left(\frac{y}{h}\right)}$$

This gives

Element 1

$$\frac{\partial\varphi_1}{\partial\left(\frac{x}{l}\right)} = qlh\,\frac{y}{h} - qlh\,\frac{y}{h}\left(\frac{x}{l}-1\right)$$

$$\frac{\partial\varphi_1}{\partial\left(\frac{y}{h}\right)} = -qlh\left(1-\frac{x}{l}\right) - \frac{qlh}{2}\left(1-\frac{x}{l}\right)^2$$

$$\sigma_x = 0$$

$$\sigma_y = \frac{1}{l^2}\frac{\partial^2 \varphi_1}{\partial\left(\frac{x}{l}\right)^2} = -\frac{1}{l^2}\, qlh\,\frac{y}{h} = -q\,\frac{y}{l}$$

$$\tau_{xy} = -\frac{1}{hl}\frac{\partial^2 \varphi_1}{\partial\left(\frac{x}{l}\right)\partial\left(\frac{y}{h}\right)} = -\frac{1}{hl}\left(qlh - qlh\left(\frac{x}{l}-1\right)\right) = q\left(\frac{x}{l}-2\right) \tag{9.61}$$

<u>Element 2</u>

$$\frac{\partial \psi_2}{\partial\left(\frac{x}{l}\right)} = qlh\,\frac{y}{h}\left(\frac{x}{l}-\frac{y}{h}\right)$$

$$\frac{\partial \psi_2}{\partial\left(\frac{y}{h}\right)} = \frac{qlh}{2}\left(\frac{y}{h}-\frac{x}{l}\right)^2 + qlh\,\frac{y}{h}\left(\frac{y}{h}-\frac{x}{l}\right)$$

$$\frac{1}{h^2}\frac{\partial^2 \psi_2}{\partial\left(\frac{y}{h}\right)^2} = \frac{ql}{h}\left(\frac{3y}{h}-2\,\frac{x}{l}\right)$$

$$\frac{1}{l^2}\frac{\partial^2 \psi_2}{\partial\left(\frac{x}{l}\right)^2} = q\,\frac{y}{l}$$

$$\frac{1}{lh}\frac{\partial^2 \psi_2}{\partial\left(\frac{x}{l}\right)\partial\left(\frac{y}{h}\right)} = -q\left(\frac{x}{l}-2\,\frac{y}{h}\right)$$

and

$$\sigma_x = 0 + \frac{ql}{h}\left(\frac{3y}{h}-2\,\frac{x}{l}\right) = \frac{ql}{h}\left(3\,\frac{y}{h}-2\,\frac{x}{l}\right)$$

$$\sigma_y = -\frac{qy}{l} + \frac{qy}{l} = 0$$

$$\tau_{xy} = q\left(\frac{x}{l}-2\right) - q\left(\frac{x}{l}-2\,\frac{y}{h}\right) = 2q\left(\frac{y}{h}-1\right) \tag{9.62}$$

Element 3

$$\frac{\partial \psi_3}{\partial \left(\frac{x}{l}\right)} = qlh\,\frac{y}{h}\left(\frac{x}{l} + \frac{y}{h}\right)$$

$$\frac{\partial \psi_3}{\partial \left(\frac{y}{h}\right)} = \frac{qlh}{2}\left(\frac{y}{h} + \frac{x}{l}\right)^2 + qlh\,\frac{y}{h}\left(\frac{y}{h} + \frac{x}{l}\right)$$

$$\frac{1}{h^2}\frac{\partial^2 \psi_3}{\partial \left(\frac{y}{h}\right)^2} = \frac{ql}{h}\left(\frac{3y}{h} + 2\,\frac{x}{l}\right)$$

$$\frac{1}{l^2}\frac{\partial^2 \psi_3}{\partial \left(\frac{x}{l}\right)^2} = q\,\frac{y}{l}$$

$$-\frac{1}{lh}\frac{\partial^2 \psi_3}{\partial \left(\frac{x}{l}\right)\partial \left(\frac{y}{h}\right)} = -q\left(\frac{x}{l} + 2\,\frac{y}{h}\right)$$

and

$$\sigma_x = 0 + \frac{ql}{h}\left(\frac{3y}{h} + 2\,\frac{x}{l}\right) = \frac{ql}{h}\left(3\,\frac{y}{h} + 2\,\frac{x}{l}\right)$$

$$\sigma_y = -\frac{qy}{h} + \frac{qy}{l} = 0$$

$$\tau_{xy} = q\left(\frac{x}{l} - 2\right) - q\left(\frac{x}{l} + 2\,\frac{y}{h}\right) = -2q\left(\frac{y}{h} + 1\right) \tag{9.63}$$

Element 4

$$\sigma_x = 0 + \frac{ql}{h}\left(\frac{3y}{h} - 2\,\frac{x}{l}\right) + \frac{ql}{h}\left(\frac{3y}{h} + \frac{2x}{l}\right) = \frac{6qly}{h^2}$$

$$\sigma_y = -\frac{qy}{l} + \frac{qy}{l} + \frac{qy}{l} = \frac{qy}{l}$$

$$\tau_{xy} = q\left(\frac{x}{l} - 2\right) - q\left(\frac{x}{l} - 2\,\frac{y}{h}\right) - q\left(\frac{x}{l} + \frac{2y}{h}\right) = -q\left(\frac{x}{l} + 2\right) \tag{9.64}$$

As a verification, at the clamping end, $x/l = -1$ and $\tau_{xy} = -q$

which is the correct value. Concerning σ_x, it is a linear function whose moment is

$$\int_{-h}^{h} \sigma_x y \, dy = \frac{6ql}{h^2} \cdot \frac{2h^3}{3} = 4qlh = (2qh) \, . \, 2l$$

This also is the correct value.

Exercise

Compute the energy $\mathcal{U}$ and the final displacement given by $v = \dfrac{\partial \mathcal{U}}{\partial T}$ where $T = 2qh$.

CHAPTER 10

SOLUTION ALGORITHM

10.1. Introduction

In chapter 3, the principle of assembling and solving the system of equations was exposed. However, such a procedure where the stiffness matrix is assembled in the central memory, is limited to very small models. Assuming that a given idealization possesses n nodes with say, 3 d.o.f. for one node, the stiffness matrix contains 9 n^2 numbers. As an example, 1000 nodes lead to 9 * 10^6 numbers. If a double precision representation of these numbers is used, this is to say 72 * 10^6 bytes ≈ 72 MB. Moreover, such a storage is uneconomical, because a large amount of terms of the stiffness matrix are zero's.

It is thus necessary to find a method where

- the stifness matrix is never assembled
- account is taken of the great amount of zero's.

These exigencies are fulfilled by the so-called frontal method, which is described in the following.

10.2. The frontal method

Consider the structure represented on fig. 65. It is possible to split it in successive substructures SS1, SS2, SS3, ... limited by successive fronts front 1, front 2, ... Let us assemble the first substructure (fig. 66). Deplacements lying on the front will be called *remaining displacements* q_R. Other displacements of the substructure may be *fixed* (q_F) or not (q_c). These last ones are called condensable. The stiffness matrix of the substructure may then be ordened in the form

$$K_{SS1} = \begin{bmatrix} K_{RR} & K_{RC} & K_{RF} \\ K_{CR} & K_{CC} & K_{CF} \\ K_{FR} & K_{FC} & K_{FF} \end{bmatrix} \tag{10.1}$$

and the corresponding force vector is

$$g_{SS1}^T = \begin{bmatrix} g_R^T & g_C^T & g_F^T \end{bmatrix} \tag{10.2}$$

Note that *except* K_{RR}, all submatrices of K_{SS1} are completely assembled and will not be modified later if the whole structure is assembled. The same is true of g_c and g_F. Consequently, it is possible *at this stage* to express the condensable d.o.f. in terms of the q_R's and the q_F's. In fact, the second line of the system

$$\begin{bmatrix} \overline{K}_{RR} & K_{RC} & K_{RF} \\ K_{CR} & K_{CC} & K_{CF} \\ K_{FR} & K_{FC} & K_{FF} \end{bmatrix} \begin{bmatrix} q_R \\ q_C \\ q_F \end{bmatrix} = \begin{bmatrix} \overline{g}_R \\ g_c \\ g_F \end{bmatrix} \tag{10.3}$$

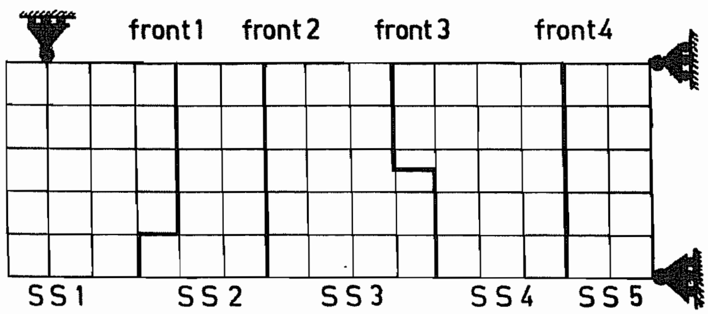

front1
front2
front3
front4
SS1
SS2
SS3
SS4
SS5

Fig. 65

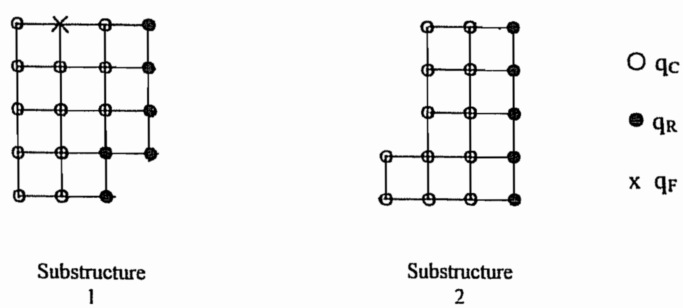

Substructure
1
Substructure
2
q_C
q_R
x q_F

Fig. 66

where incomplete blocks are overlined, is

$$K_{CR}q_R + K_{CC}q_C + K_{CF}q_F = g_C,$$

from which

$$q_C = -K_{CC}^{-1}K_{CR}q_R - K_{CC}^{-1}K_{CF}q_F + K_{CC}^{-1}g_C \tag{10.4}$$

In this relation, fixed values q_F are known. Setting this value of q_C in the first line of system (10.3) gives

$$\left(\overline{K}_{RR} - K_{RC}K_{CC}^{-1}K_{CR}\right)q_R + \left(K_{RF} - K_{RC}K_{CC}^{-1}K_{CF}\right)q_F = \overline{g}_R - K_{RC}K_{CC}^{-1}g_C$$

or

$$\left(\overline{K}_{RR} - K_{RC}K_{CC}^{-1}K_{CR}\right)q_R = \overline{g}_R - \left(K_{RF} - K_{RC}K_{CC}^{-1}K_{CF}\right)q_F - K_{RC}K_{CC}^{-1}g_C$$

it is,

$$\overline{K}_{RR}^{*}q_R = \overline{g}_R^{*} \tag{10.5}$$

with

$$\left.\begin{aligned}\overline{K}_{RR}^{*} &= \overline{K}_{RR} - K_{RC}K_{CC}^{-1}K_{CR}\\ \overline{g}_R^{*} &= \overline{g}_R - \left(K_{RF} - K_{RC}K_{CC}^{-1}K_{CF}\right)q_F - K_{RC}K_{CC}^{-1}g_C\end{aligned}\right\} \tag{10.6}$$

In the second substructure, $\overline{K}_{RR}^{*}$ and $\overline{g}_R^{*}$, as defined here, will be considered as the stiffness matrix and the load of a "superelement" which is the first substructure represented by its front. The procedure is repeated successively in subsequent substructures, and the last one, all non fixed displacements are noted q_R. This last system, of the form

$$K_{RR}q_R + K_{RF}q_F = g_R$$
$$K_{FR}q_R + K_{FF}q_F = g_F$$

may be solved. This is the end of the first stage or condensation process. We have now to *restitute* all condensed displacements by returning successively to preceeding substructure, in the reverse order. In each of them, q_C's are obtained from relations (10.4). This, of course, implies that matrices $K_{CC}^{-1}K_{CR}$ and $K_{CC}^{-1}K_{CF}$ and vectors $K_{CC}^{-1}g_C$ have been stored on a peripherical unit during the condensation process.

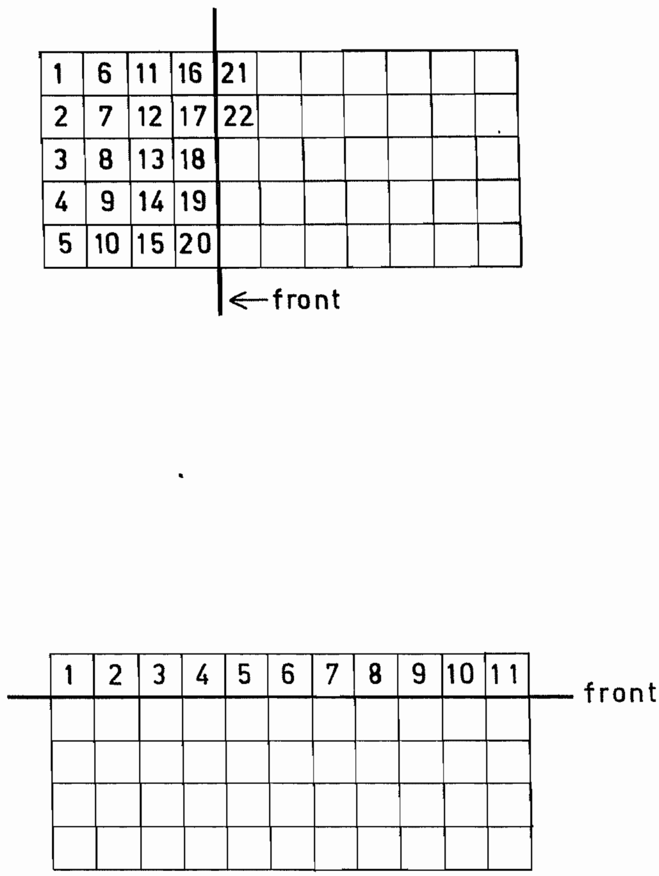

1
6
11
16
21
2
7
12
17
22
3
8
13
18
4
9
14
19
5
10
15
20
front
1
2
3
4
5
6
7
8
9
10
11
front

Fig. 67

10.3. How to cut the structure in substructures ?

The easiest way to define substructures is to assemble the *elements in their order of appearance in the element list.* This procedure has two main advantages

1) Node numbers are immaterial, being only labels
2) Element numbers also.

Beginning by the first element in the list, a pre-processor adds other elements and computes the resulting q_C's, q_R's and q_F's, stopping when there is a sufficient number of condensable displacements (e.g., 30). It also computes the total memory which is necessary to assemble successive substructures and verifies that it is compatible with the total allowed memory. This is the essential limitation of the process. It finally depends of the number of remaining displacements, it is the *front width.* As illustrated by fig. 67, the front width is the lowest when elements are presented in the direction of the least dimension. It has to be noted that in complex 3-dimensional structures, a correct choice of the order of the elements is a crucial problem in terms of feasability and, unfortunately, often a difficult problem. Special ordering programs exist for this purpose and are used at the stage of data preparation.

* *

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Appendix : The automatic monomial-based generation procedure

The following procedure, which was extensively used in the SAMCEF software, is based on a pseudo-formal treatment of polynomials. The basic idea is due to P. Beckers [7,7bis].

1. Polynomal representation

Let us consider a polynomial of the form

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 \ldots$$

It may be written as

$$u = m^T a,$$

with

$$m^T = (1, x, y, x^2, xy, y^2, \ldots)$$

$$a^T = (\alpha_1, \alpha_2, \alpha_3, \ldots).$$

The monomial matrix m^T may be represented symbolically by three vectors, namely a coefficient vector cm,

$$cm^T = (1,1,1,\ldots)$$

an x-exponent vector ixm,

$$ixm^T = (0,1,0,2,1,0,3,2,1,0,\ldots)$$

and a y-exponent vector

$$iym^T = (0,0,1,0,1,2,0,1,2,3,\ldots).$$

As is easily realized, these three vectors verify some systematic rules which may be used to generate them automatically. These vectors being generated, one has

$$u = \sum_i \alpha_i cm(i).x**ixm(i).y**iym(i).$$

2. Derivation

The derivation of a polynomial leads to two derivatives

$$\frac{\partial u}{\partial x} = \sum_i \alpha_i cmx(i).x**ixmx(i).y**iymx(i)$$

and

$$\frac{\partial u}{\partial x} = \sum_i \alpha_i cmy(i).x**ixmy(i).y**iymy(i)$$

whose coefficients and exponents are obtained by the following rules for $\frac{\partial u}{\partial x}$, say,

For each value of i,

If $ixm(i) \neq 0$, then

$$ixmx(i) = ixm(i) - 1$$
$$iymx(i) = iym(i)$$
$$cmx(i) = cm(i) * ixm(i)$$

else,

$$ixmx(i) = 0$$
$$iymx(i) = 0$$
$$cmx(i) = 0$$

endif.

Endfor

A similar procedure leads to $\frac{\partial u}{\partial y}$.

3. Strain matrix

The strain matrix B, relating the strain vector e to the parameter vector a, may be obtained by assembling vectors of the form cmx, ixmx, iymx and so on. In standard cases, each element of B is a *monomial*, so that it is possible to represent B by the three matrices CB (coefficients), IXB (x-exponents) and IYB (y-exponents), that is,

$$B(i, j) = CB(i, j).x**IXB(i, j)**IYB(i, j).$$

4. Hooke matrix

In the case of variable thickness membrane or plates, the Hooke matrix varies as a given power of the thickness. This lead to the following expansion

$$H(i,j) = \sum_k CH(i,j,k) x**IXH(i,j,k).y**IYH(i,j,k)$$

5. Integral table

One has first to give or to compute the integral table defined by

$$I(i,j) = \int_e x^{i-1} y^{j-1} dx\, dy$$

This table will make the following computations very easy.

6. Computation of the stiffness integral matrix

Recalling that

$$J_{ij} = \int_e B^T HBdS,$$

one has first

$$\left(B^T HB\right)_{ij} = \sum_{kl} B_{ki} H_{kl} B_{lj}$$

so that

$$J_{ij} = \sum_{klm} CB(k,i) CH(k,l,m) CB(l,j).I(i1,i2)$$

with

$$i1 = IXB(k,i) + IXH(k,l,m) + IXB(l,j) + 1$$

and

$$i2 = IYB(k,i) + IYH(k,l,m) + IYB(l,j) + 1.$$

As one can see, this procedure is perfectly systematic. It was the base of SAMCEF families of elements of variable degree.

In nonstandard cases, where some elements of B are no more monomials, this procedure does not work as such. However, artifices are often possible, as seen in chapter 5 with parallelograms.

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