MULTIDIMENSIONAL GENERALIZED AUTOMATIC SEQUENCES AND SHAPE-SYMMETRIC MORPHIC WORDS

EMILIE CHARLIER, TOMI KÄRKI*, AND MICHEL RIGO

Abstract. An infinite word is S-automatic if, for all $n \geq 0$, its $(n+1)$st letter is the output of a deterministic automaton fed with the representation of n in the considered numeration system S. In this paper, we consider an analogous definition in a multidimensional setting and study the relationship with the shape-symmetric infinite words as introduced by Arnaud Maes. Precisely, for $d \geq 2$, we show that a multidimensional infinite word $x : \mathbb{N}^d \to \Sigma$ over a finite alphabet Σ is S-automatic for some abstract numeration system S built on a regular language containing the empty word if and only if x is the image by a coding of a shape-symmetric infinite word.

1. Introduction

Let $k \geq 2$. An infinite word $x = (x_n)_{n\geq 0}$ is *k-automatic* if for all $n \geq 0$, x_n is obtained by feeding a deterministic finite automaton with output (DFAO for short) with the k-ary representation of n. In his seminal paper [4], A. Cobham shows that an infinite word is *k-automatic* if and only if it is the image by a coding of a fixed point of a uniform morphism of constant length k.

If we relax the assumption on the uniformity of the morphism, Cobham's result still holds but k-ary systems are replaced by a wider class of numeration systems, the so-called *abstract numeration systems* [6, 13, 12]. If an abstract numeration system is denoted by S, the corresponding sequences that can be generated are said to be *S-automatic*. That is, the $(n + 1)$st element of such a sequence is obtained by feeding a DFAO with the representation of n in the considered abstract numeration system S.

This paper studies the relationship between sequences generated by automata and sequences generated by morphisms, but extended to the framework of multidimensional infinite words, i.e., maps from $\mathbb{N}^d$ to some finite alphabet Σ. For instance, k-automatic sequences have been generalized either by considering d-tuples of k-ary representations given to a convenient DFAO or by iterating morphisms for which images of letters are d-dimensional cubes of constant size, see [14] and also [11] for questions related to frequencies of letters. In [13], multidimensional S-automatic sequences have been introduced mimicking O. Salon's construction. Let us mention [2] where a different notion of bidimensional morphisms is introduced in connection to problems arising in discrete geometry. In [5] bidimensional S-automatic sequences turn out to be useful in the context of combinatorial game theory. They play a central role to get new caracterizations of P-positions for the famous Wythoff's game and some of its variations. Another motivation for studying the set of multidimensional S-automatic words w over $\{0, 1\}$ is to consider them as characteristic words of subsets P_w of $\mathbb{N}^d$, to extend the structure $\langle \mathbb{N}; < \rangle$ by the corresponding predicates P_w and to study the decidability of the corresponding first-order theory. See also [3] for relationship with second-order monadic theory.

Our main result in this paper can be precisely stated as follows.

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Theorem. Let $d \geq 1$. The d-dimensional infinite word x is S-automatic for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ if and only if x is the image by a coding of a shape-symmetric d-dimensional infinite word.

Our first task is to present the different concepts occurring in this statement. The notion of shape-symmetry was first introduce by A. Maes and was used mainly in connection to logical questions about the decidability of first-order theories where $\langle \mathbb{N}; < \rangle$ is extended by some morphic predicate [7, 8].

1.1. **Abstract numeration systems.** If Σ is a finite alphabet, Σ^* denotes the free monoid generated by Σ having concatenation of words as product and the empty word ε as neutral element. If $w = w_0 \cdots w_{\ell-1}$ is a word, $\ell \geq 0$, where w_j's are letters, then $|w|$ denotes its length ℓ. Let $(\Sigma, <)$ be a totally ordered alphabet and u, v be two words over Σ. We say that u is *genealogically less* than v, and we write $u \prec v$ if either $|u| < |v|$ (i.e., u is of shorter length than v) or $|u| = |v|$ and there exist $p, s, t \in \Sigma^*$, $a, b \in \Sigma$ such that $u = pas$, $v = pbt$ and $a < b$ (i.e., u is lexicographically less than v). Let us also mention that we have taken the convention that all finite or infinite words and pictures have indices starting from 0.

Definition 1. An *abstract numeration system* [6] is a triple $S = (L, \Sigma, <)$ where L is an infinite regular language over a totally ordered finite alphabet $(\Sigma, <)$. Enumerating the words of L using the genealogical ordering $\prec$ induced by the ordering $<$ of Σ gives a one-to-one correspondence $\mathrm{rep}_S : \mathbb{N} \to L$ mapping the non-negative integer n onto the $(n+1)$st word in L. In particular, 0 is sent onto the first word in the genealogically ordered language L. The reciprocal map is denoted by $\mathrm{val}_S : L \to \mathbb{N}$.

Example 2. Take $\Sigma = \{a, b\}$ with $a < b$ and $L = \{a, ba\}^*\{\varepsilon, b\}$. The first words in L are ε, a, b, aa, ab, ba, aaa, aab,.... With $S = (L, \Sigma, <)$, we have for instance $\mathrm{val}_S(b) = 2$ and $\mathrm{rep}_S(5) = ba$.

Remark 3. Any positional numeration system built on a strictly increasing sequence $(U_n)_{n \geq 0}$ of integers such that $U_0 = 1$ gives an abstract numeration system whenever $\mathbb{N}$ is U-recognizable, i.e., whenever the set of greedy representations of the non-negative integers in terms of the sequence $(U_n)_{n \geq 0}$ is regular.

Any regular language is accepted by a deterministic finite automaton, which is defined as follows. A *deterministic finite automaton* $\mathcal{A}$ (DFA for short) is given by $\mathcal{A} = (Q, q_0, \Sigma, \delta, F)$ where Q is the finite *set of states*, $q_0 \in Q$ is the *initial state*, $\delta : Q \times \Sigma \to Q$ is the *transition function* and $F \subseteq Q$ is the *set of final states*. The function δ can be extended to $Q \times \Sigma^*$ by $\delta(q, \varepsilon) = q$ for all $q \in Q$ and $\delta(q, aw) = \delta(\delta(q, a), w)$ for all $q \in Q$, $a \in \Sigma$ and $w \in \Sigma^*$. A word $w \in \Sigma^*$ is *accepted by* $\mathcal{A}$ if $\delta(q_0, w) \in F$. The *langage accepted by* $\mathcal{A}$ is the set of the accepted words. A *deterministic finite automaton with output* (DFAO for short) $\mathcal{B} = (Q, q_0, \Sigma, \delta, \Gamma, \tau)$ is defined analogously where Γ is the *output alphabet* and $\tau : Q \to \Gamma$ is the *output function*. The *output corresponding to the input* $w \in \Sigma^*$ is $\tau(\delta(q_0, w))$.

1.2. **S-automatic multidimensional infinite words.** Let $d \geq 1$. To work with d-tuples of words of the same length, we introduce the following map.

Definition 4. If $w_1, \ldots, w_d$ are finite words over the alphabet Σ, the map $(\cdot)^{\#} : (\Sigma^*)^d \to ((\Sigma \cup \{\#\})^d)^*$ is defined as

$$(w_1, \ldots, w_d)^{\#} := (\#^{m-|w_1|} w_1, \ldots, \#^{m-|w_d|} w_d)$$

where $m = \max\{|w_1|, \ldots, |w_d|\}$.

As an example, $(ab, bbaa)^{\#} = (\#\#ab, bbaa)$. In what follows, we use the notation $\Sigma_{\#}$ as a shorthand for $\Sigma \cup \{\#\}$.

Definition 5. A *d-dimensional infinite word* over the alphabet Γ is a map $x : \mathbb{N}^d \to \Gamma$. We use notation like $x_{n_1,\dots,n_d}$ or $x(n_1,\dots,n_d)$ to denote the value of x at $(n_1,\dots,n_d)$. Such a word is said to be *S-automatic* if there exist an abstract numeration system $S = (L, \Sigma, <)$ and a deterministic finite automaton with output $\mathcal{A} = (Q, q_0, (\Sigma_\#)^d, \delta, \Gamma, \tau)$ such that, for all $n_1,\dots,n_d \geq 0$,

$$\tau(\delta(q_0, (\mathrm{rep}_S(n_1),\dots,\mathrm{rep}_S(n_d))^\#)) = x_{n_1,\dots,n_d}.$$

This notion was introduced in [13] (see also [10]) as a natural generalization of the multidimensional k-automatic sequences introduced in [14].

Example 6. Consider the abstract numeration system introduced in Example 2, $S = (\{a, ba\}^*\{\varepsilon, b\}, \{a, b\}, a < b)$ and the DFAO depicted in Figure 1. Since this automaton is fed with entries of the form $(\mathrm{rep}_S(n_1), \mathrm{rep}_S(n_2))^\#$, we do not consider the transitions of label $(\#, \#)$. If the outputs of the DFAO are considered to be the

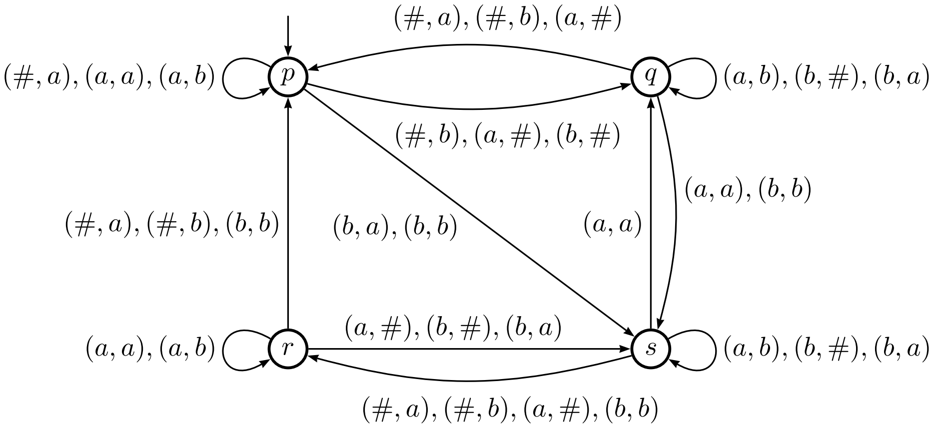

FIGURE 1. A deterministic finite automaton with output.

states themselves, then we produce the bidimensional infinite S-automatic word given in Figure 2.

	ε	a	b	aa	ab	ba	aaa	aab	$\cdots$
ε	p	q	q	p	q	p	q	q	$\cdots$
a	p	p	s	s	q	s	p	s	
b	q	p	s	q	s	q	p	s	
aa	p	p	s	p	s	q	q	s	
ab	q	p	s	p	s	s	s	r	
ba	p	s	q	p	s	q	s	q	
aaa	p	p	s	p	s	q	p	s	
aab	q	p	s	p	s	s	p	s	
$\vdots$	$\vdots$								$\ddots$

FIGURE 2. A bidimensional infinite S-automatic word.

1.3. **Multidimensional morphism.** This section is given for the sake of completeness and is mainly dedicated to present the notions of multidimensional morphism and shape-symmetry as they were introduced by A. Maes mainly in connection with the decidability question of logical theories [7, 8, 9].

If $i \leq j$ are integers, $[\![i, j]\!]$ denotes the interval of integers $\{i, i+1, \dots, j\}$. Let $d \geq 1$. If $\mathbf{n} \in \mathbb{N}^d$ and $i \in \{1, \dots, d\}$, then n_i is the ith component of $\mathbf{n}$. Let $\mathbf{m}$

and $\mathbf{n}$ be two d-tuples in $\mathbb{N}^d$. We write $\mathbf{m} \le \mathbf{n}$ (resp. $\mathbf{m} < \mathbf{n}$), if $m_i \le n_i$ (resp. $m_i < n_i$) for all $i = 1, \ldots, d$. For $\mathbf{n} \in \mathbb{N}^d$ and $j \in \mathbb{N}$, $\mathbf{n} + j := (n_1 + j, \ldots, n_d + j)$. In particular, we set $\mathbf{0} := (0, \ldots, 0)$ and $\mathbf{1} := (1, \ldots, 1)$. If $j.\mathbf{1} \le \mathbf{n}$, then we set $\mathbf{n} - j := (n_1 - j, \ldots, n_d - j)$.

Definition 7. Let $s_1, \ldots, s_d$ be positive integers or ∞. A *d-dimensional picture* over the alphabet Σ is a map x with domain $[\![0, s_1 - 1]\!] \times \cdots \times [\![0, s_d - 1]\!]$ taking values in Σ. By convention, if $s_i = \infty$ for some i, then $[\![0, s_i - 1]\!] = \mathbb{N}$. If x is such a picture, we write $|x|$ for the d-tuple $(s_1, \ldots, s_d) \in (\mathbb{N} \cup \{\infty\})^d$ which is called the *shape* of x. We denote by ε_d the d-dimensional picture of shape $(0, \ldots, 0)$. Note that $\varepsilon_1 = \varepsilon$. If $\mathbf{n} = (n_1, \ldots, n_d)$ belongs to the domain of x, we indifferently use the notation $x_{n_1,\ldots,n_d}$, $x_{\mathbf{n}}$, $x(n_1, \ldots, n_d)$ or $x(\mathbf{n})$. Let x be a d-dimensional picture. If for all $i \in \{1, \ldots, d\}$, $|x|_i < \infty$, then x is said to be *bounded.* The set of d-dimensional bounded pictures over Σ is denoted by $B_d(\Sigma)$. A bounded picture x is a *square* of *size* $c \in \mathbb{N}$ if $|x| = c.\mathbf{1}$.

Definition 8. Let x be a d-dimensional picture. If $\mathbf{0} \le \mathbf{s} \le \mathbf{t} \le |x| - 1$, then $x[\mathbf{s}, \mathbf{t}]$ is said to be a *factor* of x and is defined as the picture y of shape $\mathbf{t} - \mathbf{s} + 1$ given by $y(\mathbf{n}) = x(\mathbf{n} + \mathbf{s})$ for all $\mathbf{n} \in \mathbb{N}^d$ such that $\mathbf{n} \le \mathbf{t} - \mathbf{s}$. For any $\mathbf{u} \in \mathbb{N}^d$, the set of factors of x of shape $\mathbf{u}$ is denoted by $\mathrm{Fact}_{\mathbf{u}}(x)$.

Example 9. Consider the bidimensional (bounded) picture of shape $(5, 2)$,

$$x = \begin{array}{|c|c|c|c|c|}\hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline \end{array}\,.$$

We have

$$x[(0,0),(1,1)] = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \quad \text{and} \quad x[(2,0),(4,1)] = \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & c & d \\ \hline \end{array}\,.$$

For instance, $\mathrm{Fact}_{\mathbf{1}}(x) = \{a, b, c, d\}$ and

$$\mathrm{Fact}_{(3,2)}(x) = \left\{ \begin{array}{|c|c|c|}\hline a & b & a \\ \hline c & d & b \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline b & a & a \\ \hline d & b & c \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & c & d \\ \hline \end{array} \right\}.$$

Definition 10. Let x be a d-dimensional picture of shape $\mathbf{s} = (s_1, \ldots, s_d)$. For all $i \in \{1, \ldots, d\}$ and $k < s_i$, $x_{|i,k}$ is the $(d-1)$-dimensional picture of shape

$$|x|_{\widehat{i}} = \mathbf{s}_{\widehat{i}} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_d)$$

defined by

$$x_{|i,k}(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d) = x(n_1, \ldots, n_{i-1}, k, n_{j+1}, \ldots, n_d)$$

for all $0 \le n_j < s_j$, $j \in \{1, \ldots, d\} \setminus \{i\}$.

Definition 11. Let x, y be two d-dimensional pictures. If for some $i \in \{1, \ldots, d\}$, $|x|_{\widehat{i}} = |y|_{\widehat{i}} = (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_d)$, then we define the *concatenation* of x and y *in the direction* i as the d-dimensional picture $x \odot^i y$ of shape $(s_1, \ldots, s_{j-1}, |x|_i + |y|_i, s_{j+1}, \ldots, s_d)$ satisfying

(i) $x = (x \odot^i y)[\mathbf{0}, |x| - 1]$
(ii) $y = (x \odot^i y)[(0, \ldots, 0, |x|_i, 0, \ldots, 0), (0, \ldots, 0, |x|_i, 0, \ldots, 0) + |y| - 1]$.

The d-dimensional empty word ε_d is a word of shape $\mathbf{0}$. We extend the definition to the concatenation of ε_d and any d-dimensional word x in the direction $i \in \{1, \ldots, d\}$ by

$$\varepsilon_d \odot^i x = x \odot^i \varepsilon_d = x.$$

Especially, $\varepsilon_d \odot^i \varepsilon_d = \varepsilon_d$.

Example 12. Consider the two bidimensional pictures

$$x = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \quad \text{and} \quad y = \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & c & d \\ \hline \end{array}$$

of shape respectively $|x| = (2,2)$ and $|y| = (3,2)$. Since $|x|_{\widehat{1}} = |y|_{\widehat{1}} = 2$, we get

$$x \odot^1 y = \begin{array}{|c|c|c|c|c|}\hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline \end{array}.$$

But notice that $x \odot^2 y$ is not defined because $2 = |x|_{\widehat{2}} \neq |y|_{\widehat{2}} = 3$.

Let us now define how to erase hyperplanes from a multidimensional picture.

Definition 13. Let x be a d-dimensional picture of shape $(s_1, s_2, \ldots, s_d)$ over $\Sigma \cup \{e\}$, where e does not belong to Σ. A $(d-1)$-dimensional picture $x_{|i,k}$ is called an *e-hyperplane* of x if each letter in $x_{|i,k}$ is equal to e. *Erasing* an e-hyperplane $x_{|i,k}$ of x means replacing x with a d-dimensional picture $x' = y \odot^i z$, where

$$y = \begin{cases} x[\mathbf{0}, (s_1, \ldots, s_{i-1}, k, s_{i+1}, \ldots, s_d) - 1] & \text{if } k \geq 1, \\ \varepsilon_d & \text{otherwise,} \end{cases}$$

and

$$z = \begin{cases} x[(0, \ldots, 0, k+1, 0, \ldots, 0), |x| - 1)] & \text{if } k < s_i - 1, \\ \varepsilon_d & \text{otherwise.} \end{cases}$$

We denote by ρ_e the map which associates to any d-dimensional picture x over $\Sigma \cup \{e\}$, the picture $\rho_e(x)$ obtained by erasing iteratively every e-hyperplane of x. Moreover, we say that x is *e-erasable* if the picture $\rho_e(x)$ does not contain the letter e as a factor anymore. In other words, for each position $\mathbf{n}$ such that $x_{\mathbf{n}} = e$, there exists an integer $i \in \{1, \ldots, d\}$ such that $x_{|i,n_i}$ is an e-hyperplane.

Let x be a d-dimensional picture and $\mu : \Sigma \to B_d(\Sigma)$ be a map. Note that μ cannot necessarily be extended to a morphism on Σ^*. Indeed, if x is a picture over Σ, $\mu(x)$ is not always well defined. Depending on the shapes of the images by μ of the letters in Σ, when trying to build $\mu(x)$ by concatenating the images $\mu(x_{\mathbf{i}})$ we can obtain "holes" or "overlaps". Therefore, we introduce some restrictions on μ.

Definition 14. Let $\mu : \Sigma \to B_d(\Sigma)$ be a map and x be a d-dimensional picture such that

$$\forall i \in \{1, \ldots, d\}, \forall k < |x|_i, \forall a, b \in \mathrm{Fact}_{\mathbf{1}}(x_{|i,k}) : |\mu(a)|_i = |\mu(b)|_i.$$

Then $\mu(x)$ is defined as

$$\mu(x) = \odot^1_{0 \leq n_1 < |x|_1} \left(\cdots \left(\odot^d_{0 \leq n_d < |x|_d} \mu(x(n_1, \ldots, n_d)) \right) \right).$$

Note that the ordering of the products in the different directions is unimportant.

Example 15. Consider the map μ given by

$$a \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline b & d \\ \hline \end{array}, \; b \mapsto \begin{array}{|c|}\hline c \\ \hline b \\ \hline \end{array}, \; c \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline \end{array}, \; d \mapsto \begin{array}{|c|}\hline d \\ \hline \end{array}.$$

Let

$$x = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}.$$

Since $|\mu(a)|_2 = |\mu(b)|_2 = 2$, $|\mu(c)|_2 = |\mu(d)|_2 = 1$, $|\mu(a)|_1 = |\mu(c)|_1 = 2$ and $|\mu(b)|_1 = |\mu(d)|_1 = 1$, $\mu(x)$ is well defined and given by

$$\mu(x) = \begin{array}{|c|c|c|}\hline a & a & c \\ \hline b & d & b \\ \hline a & a & d \\ \hline \end{array}.$$

But one can notice that $\mu^2(x)$ is not well defined.

Definition 16. Let $\mu : \Sigma \to B_d(\Sigma)$ be a map. If for all $a \in \Sigma$ and all $n \geq 0$, $\mu^n(a)$ is well defined from $\mu^{n-1}(a)$, then μ is said to be a *d-dimensional morphism*.

The usual notion of a *prolongable morphism* can be given in this multidimensional setting.

Definition 17. Let μ be a d-dimensional morphism and a be a letter such that $(\mu(a))_{\mathbf{0}} = a$. We say that μ is *prolongable on a*. Then the limit

$$w = \mu^\omega(a) := \lim_{n \to +\infty} \mu^n(a)$$

is well defined and $w = \mu(w)$ is a *fixed point* of μ. A d-dimensional infinite word x over Σ is said to be *purely morphic* if it is a fixed point of a d-dimensional morphism. It is said to be *morphic* if there exists a coding $\nu : \Gamma \to \Sigma$ (i.e., a letter-to-letter morphism) such that $x = \nu(y)$ for some purely morphic word y over Γ.

The so-called property of shape-symmetry that we introduce now is a natural generalization of uniform morphisms where all images are squares of the same dimension [14].

Definition 18. Let $\mu : \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. If for any permutation f of $\{1, \ldots, d\}$ and for all $n_1, \ldots, n_d > 0$, $|\mu(x(n_{f(1)}, \ldots, n_{f(d)}))| = (s_{f(1)}, \ldots, s_{f(d)})$ whenever $|\mu(x(n_1, \ldots, n_d))| = (s_1, \ldots, s_d)$, then x is said to be *shape-symmetric* (with respect to μ).

Remark 19. An equivalent formulation of shape-symmetry is given as follows. Let $\mu : \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. This word is shape-symmetric if and only if

$$\forall i, j \leq d, \forall k \in \mathbb{N}, \forall a \in \mathrm{Fact}_{\mathbf{1}}(x_{|i,k}), \forall b \in \mathrm{Fact}_{\mathbf{1}}(x_{|j,k}) : |\mu(a)|_i = |\mu(b)|_j.$$

Remark 20. A. Maes showed that determining whether or not a map $\mu : \Sigma \to B_d(\Sigma)$ is a d-dimensional morphism is a decidable problem. Moreover he showed that if μ is prolongable on a letter a, then it is decidable whether or not the fixed point $\mu^\omega(a)$ is shape-symmetric [7, 8, 9].

Example 21. One can show that the following morphism has a fixed point $\mu^\omega(a)$ which is shape-symmetric.

$$\mu(a) = \mu(f) = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}, \ \mu(b) = \begin{array}{|c|}\hline e \\ \hline c \\ \hline\end{array}, \ \mu(c) = \begin{array}{|c|c|}\hline e & b \\ \hline\end{array}, \mu(d) = \begin{array}{|c|}\hline f \\ \hline\end{array}, \ \mu(e) = \begin{array}{|c|c|}\hline e & b \\ \hline g & d \\ \hline\end{array},$$

$$\mu(g) = \begin{array}{|c|c|}\hline h & b \\ \hline\end{array}, \ \mu(h) = \begin{array}{|c|c|}\hline h & b \\ \hline c & d \\ \hline\end{array}.$$

We have represented in Figure 3 the beginning of the picture. Some elements are underlined for the use of Example 32.

Definition 22. Let $d \geq 2$ and let $\mu : \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. The *shape sequence* of x with respect to μ *in the direction* $i \in \{1, \ldots, d\}$ is the sequence

$$\mathrm{Shape}_{\mu,i}(x) = (|\mu(x_{|i,k})|_i)_{k \geq 0}.$$

For a unidimensional morphism μ having the infinite word $x = x_0 x_1 x_2 \cdots$ as a fixed point, the *shape sequence* of x with respect to μ is $\mathrm{Shape}_\mu(x) = (|\mu(x_k)|)_{k \geq 0}$.

Remark 23. Let $\mu : \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. Note that x is shape-symmetric if and only if

$$\mathrm{Shape}_{\mu,1}(x) = \cdots = \mathrm{Shape}_{\mu,d}(x).$$

$\underline{a}$	$\underline{b}$	e	e	b	e	b	e	e	b	$\cdots$
c	d	$\underline{c}$	g	d	g	d	c	g	d	
e	b	f	e	$\underline{b}$	h	b	f	h	b	
e	b	e	a	b	e	b	e	h	b	
g	d	c	c	d	g	d	$\underline{c}$	c	d	
e	b	e	e	b	a	b	e	e	b	
g	d	c	g	d	c	d	c	g	d	
h	b	f	e	b	e	b	f	h	b	
e	b	e	e	b	e	b	e	a	b	
g	d	c	g	d	g	d	c	c	d	
$\vdots$										$\ddots$

FIGURE 3. A fixed point of μ.

2. MAIN RESULT

Let us recall that our goal is to prove the following result.

Theorem 24. *Let $d \geq 1$. The d-dimensional infinite word x is S-automatic for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ if and only if x is the image by a coding of a shape-symmetric infinite d-dimensional word.*

The case $d = 1$ is proved in [13]. It is a natural generalization of the classical Cobham's theorem from 1972 [4]. For the sake of clarity, we make the proof in the case $d = 2$. We split the proof into two parts.

Part 1. Assume that $x = \nu(\mu^\omega(a))$ where $\nu : \Sigma \to \Gamma$ is a coding and $\mu : \Sigma \to B_2(\Sigma)$ is a 2-dimensional morphism prolongable on a such that $y = \mu^\omega(a)$ is shape-symmetric. We show in this part that x is S-automatic for some $S = (L, \Sigma, <)$ where $\varepsilon \in L$.

Let $Y_1 = (y_{n,0})_{n\geq 0}$ be the first line of y. This word Y_1 is a unidimensional infinite word over a subset Σ_1 of Σ. It is clear that Y_1 is generated by a unidimensional morphism μ_1 derived from μ (one has only to consider the first line occurring in the images by μ of the letters in Σ).

Definition 25. With each (unidimensional) morphism $\mu : \Sigma \to \Sigma^*$ and with each letter $a \in \Sigma$ we can canonically associate a DFA denoted by $\mathcal{A}_{\mu,a}$ and defined as follows. Let $r_\mu := \max_{b\in\Sigma} |\mu(b)|$. The alphabet of $\mathcal{A}_{\mu,a}$ is $\{0, \ldots, r_\mu - 1\}$. The set of states is Σ. The initial state is a and every state is final. The (partial) transition function δ_μ is defined by $\delta_\mu(b, i) = (\mu(b))_i$, for all $b \in \Sigma$ and $i \in \{0, \ldots, |\mu(b)| - 1\}$. The language accepted by $\mathcal{A}_{\mu,a}$ from which are removed the words having 0 as a prefix is called the *directive language* of (μ, a) and is denoted by $L_{\mu,a}$. Note that $L_{\mu,a}$ is a prefix language since all states in $\mathcal{A}_{\mu,a}$ are final. In particular, we have $\varepsilon \in L_{\mu,a}$. The reason why we call it *directive* will be clear, see Lemma 27 and Lemma 28.

Example 26. Considering Example 21, $\Sigma_1 = \{a, b, e\}$, $\mu_1 : a \mapsto ab,\ b \mapsto e,\ e \mapsto eb$ and $Y_1 = abeebebeebeebebeebebeeb\cdots$. The DFA associated with (μ_1, a) is depicted in Figure 4. The first words in the directive language of (μ_1, a) are

$$L_{\mu_1,a} = \{\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, \ldots\}.$$

Lemma 27. *Let $\mu : \Sigma \to \Sigma^*$ be a morphism prolongable on $a \in \Sigma$. Let S be the abstract numeration system built on the directive language $L_{\mu,a}$ of (μ, a) with the*

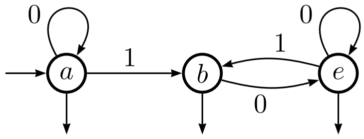

FIGURE 4. The automaton $\mathcal{A}_{\mu_1,a}$.

ordered alphabet $(\{0,\dots,r_\mu-1\}, 0<\dots<r_\mu-1)$. *Then, for the infinite word* $\mu^\omega(a)=y_0y_1y_2\cdots$ *and for all* $n\ge 0$, *we have*

$$y_n=\delta_\mu(a,\operatorname{rep}_S(n))$$

and

$$\mu(y_n)=\mu^\omega(a)[\operatorname{val}_S(\operatorname{rep}_S(n)0),\operatorname{val}_S(\operatorname{rep}_S(n)(|\mu(y_n)|-1))].$$

Proof. The adjacency matrix $M\in\mathbb{N}^{\Sigma\times\Sigma}$ of $\mathcal{A}_{\mu,a}$ is defined for all $b,c\in\Sigma$ by $M_{b,c}=\#\{i:\delta_\mu(b,i)=c\}$. For all $s>0$, $[M^s]_{b,c}$ is the number of paths of length s from b to c in $\mathcal{A}_{\mu,a}$. Since all states are final, the number N_s of words of length s accepted by $\mathcal{A}_{\mu,a}$ is obtained by summing up all the entries of M^s in the row corresponding to a. Because $\mathcal{A}_{\mu,a}$ has a loop of label 0 in a, the number of words of length s accepted by $\mathcal{A}_{\mu,a}$ and starting with 0 is equal to the number N_{s-1} of words of length $s-1$ accepted by $\mathcal{A}_{\mu,a}$. Consequently, the number of words of length s in the directive language $L_{\mu,a}$ is exactly N_s-N_{s-1}. Of course, the matrix M can also be related to the morphism μ and $M_{b,c}$ is also the number of occurrences of c in $\mu(b)$. In particular, summing up all entries in the row of M^s corresponding to a gives $|\mu^s(a)|$. Therefore, the number of words of length s in the directive language $L_{\mu,a}$ is $|\mu^s(a)|-|\mu^{s-1}(a)|$ and we get that

$$|\operatorname{rep}_S(n)|=s\Leftrightarrow n\in\{|\mu^{s-1}(a)|,\dots,|\mu^s(a)|-1\}. \tag{1}$$

In particular, if $0<n<|\mu(a)|$, we have $|\operatorname{rep}_S(n)|=1$ and in this case $\operatorname{rep}_S(n)=n$. Since we have $\operatorname{rep}_S(0)=\varepsilon$ and $\mu(a)=au$, for some $u\in\Sigma^*$, we get $y_0=a=\delta_\mu(a,\operatorname{rep}_S(0))$. Hence, by the definition of $\mathcal{A}_{\mu,a}$, we have that $y_n=\delta_\mu(a,\operatorname{rep}_S(n))$ for $n<|\mu(a)|$. Now let $s>0$ and assume that $y_n=\delta_\mu(a,\operatorname{rep}_S(n))$ for all $n<|\mu^s(a)|$. Let $|\mu^s(a)|\le n<|\mu^{s+1}(a)|$. There exist a unique $|\mu^{s-1}(a)|\le m<|\mu^s(a)|$ such that

$$\mu^{s+1}(a)=\underbrace{\mu^{s-1}(a)uy_mv}_{\mu^s(a)}\mu(u)\underbrace{xy_ny}_{\mu(y_m)}\mu(v),$$

for some words u,v,x,z. Therefore $y_n=(\mu(y_m))_i$ for some $i\in\{0,\dots,|\mu(y_m)|-1\}$. Then by the definition of $\mathcal{A}_{\mu,a}$, we have

$$y_n=\delta_\mu(y_m,i)=\delta_\mu(\delta_\mu(a,\operatorname{rep}_S(m)),i)=\delta_\mu(a,\operatorname{rep}_S(m)i)$$

and in view of condition (1) and again by the definition of $\mathcal{A}_{\mu,a}$, we get

$$\operatorname{val}_S(\operatorname{rep}_S(m)i)=|\mu^s(a)|+|\mu(y_{|\mu^{s-1}(a)|})|+\dots+|\mu(y_{m-1})|+i=n.$$

Hence, $\operatorname{rep}_S(n)=\operatorname{rep}_S(m)i$ and the result follows. □

The following lemma is simply another formulation of the previous result.

Lemma 28. *Let* $\mu:\Sigma\to\Sigma^*$ *be a morphism prolongable on* $a\in\Sigma$ *and let* $\mu^\omega(a)=y_0y_1y_2\cdots$. *Let* S *be the abstract numeration system built on the directive language* $L_{\mu,a}$ *of* (μ,a) *with the ordered alphabet* $(\{0,\dots,r_\mu-1\},0<\dots<r_\mu-1)$. *Let* $n\ge 0$ *and* $\operatorname{rep}_S(n)=w_0\cdots w_\ell$, *where* w_j*'s are letters. Define* $z_0:=\mu(a)$ *and for* $j=0,\dots,\ell-1$, *set* $z_{j+1}:=\mu((z_j)_{w_j})$. *Then,* $y_n=(z_\ell)_{w_\ell}$.

Example 29. Continue Example 26. The fixed point Y_1 of μ_1 start with

$$abeebebe = y_0 \cdots y_7$$

and $\text{rep}_S(7) = 1010$. From Lemma 27, $y_7 = e$ has been generated applying μ_1 to the letter in the position $\text{val}_S(101) = 4$, i.e., $y_4 = b$. We have $y_7 = (\mu_1(b))_0$. In turn, y_4 occurs in the image by μ_1 of the letter in the position $\text{val}_S(10) = 2$, $y_2 = e$ and we have $y_4 = (\mu_1(e))_1$. Now y_2 appears in the image of the letter in the position $\text{val}_S(1) = 1$ and we have $y_2 = (\mu_1(b))_0$.

The following result is obvious.

Lemma 30. *Let x, y be two infinite (unidimensional) words and λ, μ be two morphisms such that there exist letters a, b such that $x = \lambda^\omega(a)$ and $y = \mu^\omega(b)$. The languages $L_{\lambda,a}$ and $L_{\mu,b}$ are equal if and only if* $\text{Shape}_\lambda(x) = \text{Shape}_\mu(y)$.

Example 31. If one considers the morphism μ_2 defined by $a \mapsto ac$, $c \mapsto e$, $e \mapsto eg$, $g \mapsto h$ and $h \mapsto hc$ (which is derived from the first column of the bidimensional morphism in Example 21), we have the DFA $\mathcal{A}_{\mu_2,a}$ depicted in Figure 5. The

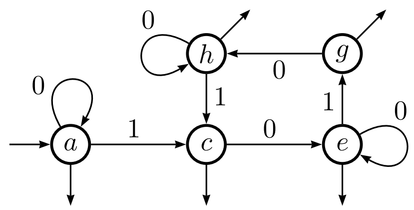

FIGURE 5. The automaton $\mathcal{A}_{\mu_2,a}$.

automata in Figure 4 and Figure 5 clearly accept the same language (the first one being minimal).

Let $Y_2 = (y_{1,n})_{n\geq 0}$ be the first column of y. This word Y_2 is a unidimensional infinite word over a subset Σ_2 of Σ. It is clear that Y_2 is generated by a morphism μ_2 derived from μ. Since y is shape-symmetric, thanks to Remark 23 and to Lemma 30, we have

$$L_{\mu_1,a} = L_{\mu_2,a} =: L.$$

We consider the abstract numeration system built upon this language L (with the natural ordering of digits). With all the above discussion and in particular in view of Lemma 28, it is clear that if $\text{rep}_S(m) = ub$, $\text{rep}_S(n) = vc$ where b, c are letters, then

$$(\mu(y_{\text{val}_S(u),\text{val}_S(v)}))_{b,c} = y_{m,n}. \tag{2}$$

Example 32. Consider the letter c occurring in the position $(7, 4)$ in the fixed point y of μ underlined in Figure 3. We have $(7, 4) = (\text{val}_S(1010), \text{val}_S(101))$. If we consider the pair $(\text{val}_S(101), \text{val}_S(10)) = (4, 2)$, we get $(\mu(y_{4,2}))_{0,1} = (\mu(b))_{0,1} = c = y_{7,4}$. In other words, $y_{7,4}$ comes from $y_{4,2}$. We can continue this way. We have $b = y_{4,2} = (\mu(y_{2,1}))_{1,0}$ because $(\text{val}_S(10), \text{val}_S(1)) = (2, 1)$. Now $y_{2,1} = c = (\mu(y_{1,0}))_{0,1}$ because $(\text{val}_S(1), \text{val}_S(\varepsilon)) = (1, 0)$. Finally $y_{1,0} = b = (\mu(y_{0,0}))_{1,0} = (\mu(a))_{1,0}$ because $(\text{val}_S(\varepsilon), \text{val}_S(\varepsilon)) = (0, 0)$.

We now extend Definition 25 to the multidimensional case.

Definition 33. For each d-dimensional morphism $\mu \colon \Sigma \to B_d(\Sigma)$ and for each letter $a \in \Sigma$, define a DFA $\mathcal{A}_{\mu,a}$ over the alphabet $\{0, \ldots, r_\mu - 1\}^d$ where $r_\mu = \max\{|\mu(b)|_i : b \in \Sigma, i = 1, \ldots, d\}$. The set of states is Σ, the initial state is a and all states are final. The (partial) transition function is defined by

$$\delta_\mu(b, \mathbf{n}) = (\mu(b))_\mathbf{n},$$

for all $b \in \Sigma$ and $\mathbf{n} \leq |\mu(b)|$.

Thanks to (2), the automaton $\mathcal{A}_{\mu,a}$ is such that, for all $m, n \geq 0$,

$$y_{m,n} = \delta_\mu(a, (\text{rep}_S(m), \text{rep}_S(n))^0),$$

where we have padded the shortest word with enough 0's to make two words of the same length as in Definition 4. If we consider the coding ν as the output function, the corresponding DFAO generates x as an S-automatic sequence. Note that padding with 0's works correctly since 0 is the lexicographically smallest letter and the directive language L does not contain any words starting with 0. This concludes the first part.

Example 34. Consider the 2-dimensional morphism μ of Example 21 and its fixed point $\mu^\omega(a)$ depicted in Figure 3. If $S = (L, \{0,1\}, 0 < 1)$ is the abstract numeration system constructed on $L = \{\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, \ldots\}$, then the corresponding DFAO depicted in Figure 6, where the output function is the identity, generates $\mu^\omega(a)$ as an S-automatic word. For instance, if we continue Example 32, by reading $(\text{rep}_S(7), \text{rep}_S(4))^0 = (1010, 0101)$, we get

$$y_{0,0} = a \overset{(1,0)}{\rightarrow} y_{1,0} = b \overset{(0,1)}{\rightarrow} y_{2,1} = c \overset{(1,0)}{\rightarrow} y_{4,2} = b \overset{(0,1)}{\rightarrow} y_{7,4} = c,$$

and the letters appearing in this sequence of transitions are exactly the underlined ones in Figure 3.

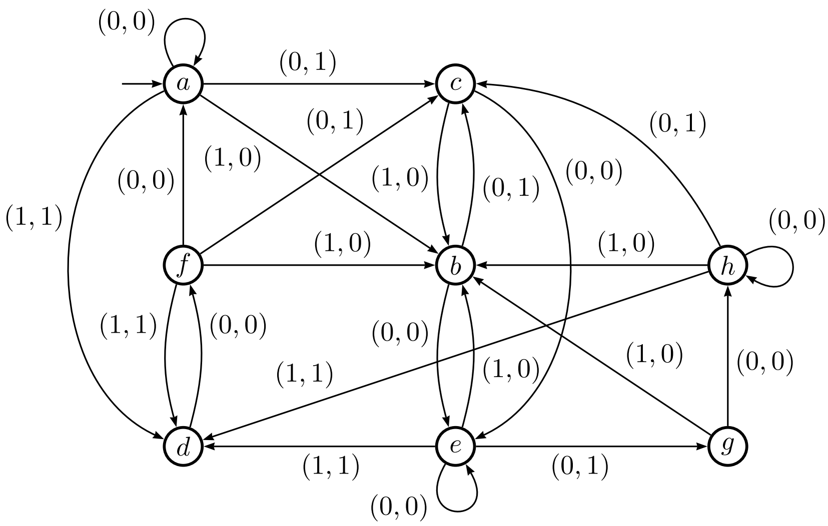

FIGURE 6. DFAO generating $\mu^\omega(a)$ as an S-automatic word.

Part 2. Assume that $x = (x_{m,n})_{m,n \geq 0}$ is a 2-dimensional S-automatic infinite word over Γ for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ and $\Sigma = \{a_1, \ldots, a_r\}$ with $a_1 < \cdots < a_r$. Let $\mathcal{A} = (Q_\mathcal{A}, q_0, (\Sigma_\#)^2, \delta_\mathcal{A}, \Gamma, \tau_\mathcal{A})$ be a deterministic finite automaton with output generating x where we may assume that $\# =: a_0$ is a symbol not belonging to Σ and that $a_0 < a_1$. Recall that this means that $x_{m,n} = \tau_\mathcal{A}(\delta_\mathcal{A}(q_0, (\text{rep}_S(m), \text{rep}_S(n))^\#))$ for all $m, n \geq 0$. Without loss of generality, we suppose that $\delta_\mathcal{A}(q, (\#, \#)) = q$, for all $q \in Q_\mathcal{A}$. In this part we prove that x can be represented as the image by a coding of a morphic shape-symmetric 2-dimensional infinite word. We do the proof in three steps. First, we show that x can be obtained applying an erasing map to a fixed point of a uniform 2-dimensional morphism. In the second step we prove that x is morphic. The generating morphism μ and the coding ν are obtained using a construction represented for dimension one in [1]. Finally, we show that the considered fixed point of μ is shape-symmetric.

Definition 35. Let $d \geq 1$. Any DFA of the form $\mathcal{A} = (Q, q_0, \Sigma^d, \delta, F)$, where $\Sigma = \{a_0, a_1, \dots, a_r\}$ with the ordering $a_i < a_{i+1}$ for all $0 \leq i \leq r-1$, can be canonically associated with a d-dimensional morphism denoted by $\mu_{\mathcal{A}} \colon Q \to B_d(Q)$ and defined as follows. The image of a letter $q \in Q$ is a d-dimensional square x of size $r+1$ defined by $x_{\mathbf{n}} = \delta(q, (a_{n_1}, \cdots, a_{n_d}))$, for all $\mathbf{0} \leq \mathbf{n} = (n_1, \dots, n_d) \leq r.\mathbf{1}$.

Example 36. Consider the alphabet $\Sigma = \{\#, a, b\}$ with $\# < a < b$ and the automaton $\mathcal{A}$ depicted in Figure 1 with added loops of label $(\#, \#)$ on all states. Then we get

$$\mu_{\mathcal{A}}(p) = \begin{array}{|c|c|c|}\hline p & q & q \\ \hline p & p & s \\ \hline q & p & s \\ \hline \end{array}\,, \mu_{\mathcal{A}}(q) = \begin{array}{|c|c|c|}\hline q & p & q \\ \hline p & s & q \\ \hline p & q & s \\ \hline \end{array}\,, \mu_{\mathcal{A}}(r) = \begin{array}{|c|c|c|}\hline r & s & s \\ \hline p & r & s \\ \hline p & r & p \\ \hline \end{array}\,, \mu_{\mathcal{A}}(s) = \begin{array}{|c|c|c|}\hline s & r & s \\ \hline r & q & s \\ \hline r & s & r \\ \hline \end{array}$$

and $\mu_{\mathcal{A}}{}^{\omega}(p)$ is the 2-dimensional infinite word depicted in Figure 7. Notice that $\mu_{\mathcal{A}}{}^{\omega}(p)$ is different from the S-automatic word given in Figure 2. However, by erasing some rows and columns in Figure 7, we obtain exactly the word in Figure 2.

p	q	q	q	p	q	q	p	q	q	p	$\cdots$
p	p	s	p	s	q	p	s	q	p	s	
q	p	s	p	q	s	p	q	s	p	q	
p	q	q	p	q	q	s	r	s	p	q	
p	p	s	p	p	s	r	q	s	p	p	
q	p	s	q	p	s	r	s	r	q	p	
q	p	q	p	q	q	s	r	s	p	q	
p	s	q	p	p	s	r	q	s	p	p	
p	q	s	q	p	s	r	s	r	q	p	
p	q	q	q	p	q	q	p	q	p	q	
p	p	s	p	s	q	p	s	q	p	p	
$\vdots$											$\ddots$

FIGURE 7. The fixed point $\mu_{\mathcal{A}}{}^{\omega}(p)$.

By assumption, L is a regular language over Σ. Hence, there exists a DFA accepting L and we may easily modify it to obtain a DFA $\mathcal{L} = (Q_{\mathcal{L}}, \ell_0, \Sigma_\#, \delta_{\mathcal{L}}, F_{\mathcal{L}})$ accepting $\{\#\}^*L$ and satisfying $\delta_{\mathcal{L}}(l_0, \#) = l_0$. Note that l_0 is a final state since $\varepsilon \in L$. Let us next define a "product" automaton $\mathcal{P} = (Q, p_0, (\Sigma_\#)^2, \delta, F)$ imitating the behavior of $\mathcal{A}$ and two copies of the automaton $\mathcal{L}$, one for each dimension. The set of states of $\mathcal{P}$ is the Cartesian product $Q = Q_{\mathcal{A}} \times Q_{\mathcal{L}} \times Q_{\mathcal{L}}$, where the initial state p_0 is (q_0, ℓ_0, ℓ_0). The transition function $\delta \colon Q \times (\Sigma_\#)^2 \to Q$ is defined by

$$\delta((q, k, \ell), (a, b)) = (\delta_{\mathcal{A}}(q, (a, b)), \delta_{\mathcal{L}}(k, a), \delta_{\mathcal{L}}(\ell, b)),$$

where (q, k, ℓ) belongs to Q and (a, b) is a pair of letters in $(\Sigma_\#)^2$. The set of final states is $F = Q_{\mathcal{A}} \times F_{\mathcal{L}} \times F_{\mathcal{L}}$. Let $y = (y_{m,n})_{m,n \geq 0}$ be the infinite word satisfying

$$y_{m,n} = \delta(p_0, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^\#).$$

Note that both the first and the second component of $(\mathrm{rep}_S(m), \mathrm{rep}_S(n))^\#$ belong to the language $\{\#\}^*L$ and, therefore, $\delta(p_0, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^\#)$ is a final state.

Define $\tau\colon F \to \Gamma$ to be the coding satisfying $\tau((q,k,\ell)) = \tau_{\mathcal{A}}(q)$ for all $(q,k,\ell) \in F$. By construction, it is clear that $\tau(y) = (x_{m,n})_{m,n\geq 0}$. We consider the canonically associated morphism $\mu_{\mathcal{P}}\colon Q \to \mathcal{B}_2(Q)$ given in Definition 35. Note that $\mu_{\mathcal{P}}$ is prolongable on p_0, since $\delta(p_0,(a_0,a_0)) = (\delta_A(q_0,(\#,\#)), \delta_{\mathcal{L}}(l_0,\#), \delta_{\mathcal{L}}(l_0,\#)) = (q_0,l_0,l_0) = p_0$. Moreover, ${\mu_{\mathcal{P}}}^{\omega}(p_0)$ is shape-symmetric with respect to $\mu_{\mathcal{P}}$, since $\mu_{\mathcal{P}}(q)$ is a square of size $r+1$ for all $q \in Q$.

Example 37. Let us continue Example 6 and consider again the abstract numeration system $S = (\{a,ba\}^*\{\varepsilon,b\},\{a,b\},a<b)$ and the DFAO depicted in Figure 1, with additional loops of label $(\#,\#)$ on all states. The minimal automaton of $\{\#\}^*\{a,ba\}^*\{\varepsilon,b\}$ is depicted in Figure 8. If $\mathcal{P}$ is the corresponding product au-

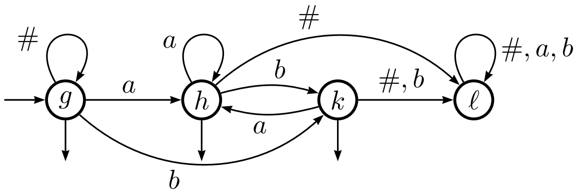

FIGURE 8. The minimal automaton accepting $\{\#\}^*\{a,ba\}^*\{\varepsilon,b\}$.

tomaton, then the fixed point ${\mu_{\mathcal{P}}}^{\omega}((p,g,g))$ of $\mu_{\mathcal{P}}$ is the 2-dimensional infinite word depicted in Figure 9.

(p,g,g)	(q,h,g)	(q,k,g)	(q,ℓ,g)	(p,h,g)	(q,k,g)	(q,ℓ,g)	(p,h,g)	$\cdots$
(p,g,h)	(p,h,h)	(s,k,h)	(p,ℓ,h)	(s,h,h)	(q,k,h)	(p,ℓ,h)	(s,h,h)	
(q,g,k)	(p,h,k)	(s,k,k)	(p,ℓ,k)	(q,h,k)	(s,k,k)	(p,ℓ,k)	(q,h,k)	
(p,g,ℓ)	(q,h,ℓ)	(q,k,ℓ)	(p,ℓ,ℓ)	(q,h,ℓ)	(q,k,ℓ)	(s,ℓ,ℓ)	(r,h,ℓ)	
(p,g,h)	(p,h,h)	(s,k,h)	(p,ℓ,h)	(p,h,h)	(s,k,h)	(r,ℓ,h)	(q,h,h)	
(q,g,k)	(p,h,k)	(s,k,k)	(q,ℓ,k)	(p,h,k)	(s,k,k)	(r,ℓ,k)	(s,h,k)	
(q,g,ℓ)	(p,h,ℓ)	(q,k,ℓ)	(p,ℓ,ℓ)	(q,h,ℓ)	(q,k,ℓ)	(s,ℓ,ℓ)	(r,h,ℓ)	
(p,g,h)	(s,h,h)	(q,k,h)	(p,ℓ,h)	(p,h,h)	(s,k,h)	(r,ℓ,h)	(q,h,h)	
(p,g,ℓ)	(p,h,ℓ)	(s,k,ℓ)	(p,ℓ,ℓ)	(p,h,ℓ)	(s,k,ℓ)	(r,ℓ,ℓ)	(s,h,ℓ)	
(p,g,ℓ)	(q,h,ℓ)	(q,k,ℓ)	(q,ℓ,ℓ)	(p,h,ℓ)	(q,k,ℓ)	(q,ℓ,ℓ)	(p,h,ℓ)	
(p,g,ℓ)	(p,h,ℓ)	(s,k,ℓ)	(p,ℓ,ℓ)	(s,h,ℓ)	(q,k,ℓ)	(p,ℓ,ℓ)	(s,h,ℓ)	
$\vdots$								$\ddots$

FIGURE 9. The fixed point ${\mu_{\mathcal{P}}}^{\omega}((p,g,g))$.

Let e be a new symbol. Recall that ρ_e is the erasing map given in Definition 13. Denote $\rho = \rho_e \circ \lambda$, where λ is a morphism on $Q \cup \{e\}$ defined by

$$\lambda(p) = \begin{cases} e & \text{if } p \notin F, \\ p & \text{otherwise.} \end{cases}$$

We claim that $y = \rho({\mu_{\mathcal{P}}}^{\omega}(p_0))$. Observe that the infinite word $\lambda({\mu_{\mathcal{P}}}^{\omega}(p_0))$ is e-erasable. Namely, all letters in a fixed column C of the infinite bidimensional word ${\mu_{\mathcal{P}}}^{\omega}(p_0)$ are of the form (q,k,ℓ) where the second component k is fixed. If k does not belong to $F_{\mathcal{L}}$, the word $\lambda(C)$ is a unidimensional e-hyperplane of $\lambda({\mu_{\mathcal{P}}}^{\omega}(p_0))$. Thus, the map ρ erases all columns where the second component k does not belong to $F_{\mathcal{L}}$. The same holds for rows and third components ℓ of the letters in Q. Hence, the 2-dimensional infinite word $\rho({\mu_{\mathcal{P}}}^{\omega}(p_0))$ contains only letters belonging to F. By the construction of the morphism $\mu_{\mathcal{P}}$, those letters are coming from the automaton $\mathcal{P}$ by feeding it with words belonging to $((\Sigma_\#)^2)^* \cap (\{\#\}^*L)^2$. More precisely, all rows and columns not belonging to y are erased and $(\rho(\mu_{\mathcal{P}}{}^{\omega}(p_0)))_{m,n}$ is equal to $\delta(p_0, (\text{rep}_S(m), \text{rep}_S(n))^{\#}) = y_{m,n}$. Hence, defining $\vartheta = \tau \circ \rho$, we get a map from Σ to Γ such that $x = \vartheta(\mu_{\mathcal{P}}{}^{\omega}(p_0))$.

Example 38. We continue Example 37 and we consider this time the bidimensional infinite S-automatic word depicted in Figure 2. This word is exactly the 2-dimensional infinite word obtained by first erasing all columns with ℓ as the second component and all rows with ℓ as the third component from the 2-dimensional infinite word $\mu_{\mathcal{P}}{}^{\omega}((p, g, g))$ depicted in Figure 9 and then mapping the infinite word by τ.

Next we show that x is morphic by getting rid of the erasing map ρ. We construct a morphism μ prolongable on some letter α and a coding ν such that $x = \nu(\mu^{\omega}(\alpha))$. We follow the guidelines of [1, Theorem 7.7.4]. First we need the following definitions.

Definition 39. Let μ be a morphism on some finite alphabet Σ and let $\Psi \subseteq \Sigma$. We say that a letter $a \in \Sigma$ is

(i) (μ, Ψ)-*dead* if the word $\mu^n(a) \in \Psi^*$ for every $n \geq 0$.
(ii) (μ, Ψ)-*moribund* if there exists $m \geq 0$ such that the word $\mu^m(a)$ contains at least one letter in $\Sigma \setminus \Psi$, and for every $n > m$, $\mu^n(a) \in \Psi^*$.
(iii) (μ, Ψ)-*robust* if there exist infinitely many $n \geq 0$ such that the word $\mu^n(a)$ contains at least one letter in $\Sigma \setminus \Psi$.

The following lemma from [1, Lemma 7.7.3] is valid also for multidimensional morphisms, since the proof is only based on the finiteness of the alphabet Σ.

Lemma 40. *Let μ be a morphism on some finite alphabet Σ and let $\Psi \subseteq \Sigma$. Then there exists an integer $T \geq 1$ such that the morphism $\varphi = \mu^T$ satisfies:*

(a) *If a is (φ, Ψ)-moribund, then $\varphi^n(a) \in \Psi^*$ for all $n > 0$ and $a \in \Sigma \setminus \Psi$.*
(b) *If a is (φ, Ψ)-robust, then the word $\varphi^n(a)$ contains at least one letter in $\Sigma \setminus \Psi$ for all $n > 0$.*

Remark 41. Note that by Lemma 40 a letter in Ψ is either (φ, Ψ)-dead or (φ, Ψ)-robust and a letter in $\Sigma \setminus \Psi$ is either (φ, Ψ)-moribund or (φ, Ψ)-robust.

We may assume, by taking a power of $\mu_{\mathcal{P}}$ if necessary, that $\mu_{\mathcal{P}}$ satisfies the properties (a) and (b) listed for φ in Lemma 40 with $\Psi = F^c := Q \setminus F$. For the sake of simplicity, we use the words *dead*, *moribund* and *robust* instead of $(\mu_{\mathcal{P}}, F^c)$-dead, $(\mu_{\mathcal{P}}, F^c)$-moribund and $(\mu_{\mathcal{P}}, F^c)$-robust from now on.

Next we classify the states of $Q_{\mathcal{L}}$ and Q into four categories. The *type of a state* $k \in Q_{\mathcal{L}}$ is

$$T_k = \begin{cases} \Delta & \text{if } k \notin F_{\mathcal{L}} \text{ and } \delta_{\mathcal{L}}(k, a) \notin F_{\mathcal{L}} \text{ for every } a \in \Sigma_{\#}, \\ M & \text{if } k \in F_{\mathcal{L}} \text{ and } \delta_{\mathcal{L}}(k, a) \notin F_{\mathcal{L}} \text{ for every } a \in \Sigma_{\#}, \\ R_{F^c} & \text{if } k \notin F_{\mathcal{L}} \text{ and there exists a letter } a \in \Sigma_{\#} \text{ such that } \delta_{\mathcal{L}}(k, a) \in F_{\mathcal{L}}, \\ R_F & \text{if } k \in F_{\mathcal{L}} \text{ and there exists a letter } a \in \Sigma_{\#} \text{ such that } \delta_{\mathcal{L}}(k, a) \in F_{\mathcal{L}}. \end{cases}$$

The *type of a state* $p = (q, k, \ell) \in Q$ is

$$T_p = \begin{cases} \Delta & \text{if } p \text{ is dead}, \\ M & \text{if } p \text{ is moribund}, \\ R_{F^c} & \text{if } p \in F^c \text{ and } p \text{ is robust}, \\ R_F & \text{if } p \in F \text{ and } p \text{ is robust}. \end{cases}$$

By these definitions, it is clear that the type of $(q, k, \ell) \in Q$ only depends on the types of k and $\ell \in Q_{\mathcal{L}}$ according to Figure 10. Note that by the properties (a) and (b) of Lemma 40, it suffices to consider transitions $\delta_{\mathcal{L}}(k, a)$ by each letter $a \in \Sigma_{\#}$

instead of transitions $\delta_{\mathcal{L}}(k, w)$ by all words w in $(\Sigma_{\#})^*$. For instance, if the type of k is R_{F^c} and the type of ℓ is R_F, then $k \notin F_{\mathcal{L}}$ and (q, k, ℓ) belongs to F^c. Moreover, there exist $m, n \in [\![0, r]\!]$ such that $\delta_{\mathcal{L}}(k, a_m) \in F_{\mathcal{L}}$ and $\delta_{\mathcal{L}}(\ell, a_n) \in F_{\mathcal{L}}$. This means that $(\mu_{\mathcal{P}}((q, k, \ell)))_{m,n}$ belongs to F. Hence, by Lemma 40 and Remark 41, (q, k, ℓ) is robust.

T_ℓ \ T_k	Δ	M	R_{F^c}	R_F
Δ	Δ	Δ	Δ	Δ
M	Δ	M	Δ	M
R_{F^c}	Δ	Δ	R_{F^c}	R_{F^c}
R_F	Δ	M	R_{F^c}	R_F

FIGURE 10. Type T_p of a letter $p = (q, k, \ell) \in Q$.

Let us define two morphisms λ_Δ and λ_M on $Q \cup \{e\}$ in a similar way as λ was defined above :

$$\lambda_\Delta(p) = \begin{cases} e & \text{if } p \text{ is dead,} \\ p & \text{otherwise;} \end{cases}$$
$$\lambda_M(p) = \begin{cases} e & \text{if } p \text{ is moribund,} \\ p & \text{otherwise.} \end{cases}$$

By the property (b) of Lemma 40, we know that if p is robust, then $\mu_{\mathcal{P}}(p)$ contains at least one letter in F and since every dead letter must belong to F^c, the word $\lambda_\Delta(\mu_{\mathcal{P}}(p))$ contains at least one letter in F. For any $\ell \in Q_{\mathcal{L}}$, let us define a sequence $(d_\ell(i))_{0 \le i \le h_\ell}$ such that $d_\ell(0) = 0$, $d_\ell(h_\ell) = r+1$ and for all $i \in [\![0, h_\ell - 1]\!]$, $d_\ell(i) < d_\ell(i+1)$ and there exists exactly one index $n \in [\![d_\ell(i), d_\ell(i+1)-1]\!]$ satisfying

$$\delta_{\mathcal{L}}(\ell, a_n) \in F_{\mathcal{L}}. \tag{3}$$

Note that h_ℓ is the number of letters $a_n \in \Sigma_{\#}$ satisfying condition (3). Hence, for each robust letter $p = (q, k, \ell)$, we get $h_k, h_\ell \ge 1$ and we may define the factorization

$$\lambda_\Delta(\mu_{\mathcal{P}}(p)) = \boxed{\begin{matrix} w_p(0,0) & w_p(1,0) & \cdots & w_p(h_k-1,0) \\ w_p(0,1) & w_p(1,1) & \cdots & w_p(h_k-1,1) \\ \vdots & \vdots & \ddots & \vdots \\ w_p(0,h_\ell-1) & w_p(1,h_\ell-1) & \cdots & w_p(h_k-1,h_\ell-1) \end{matrix}},$$

where each bidimensional picture

$$w_p(i,j) = \lambda_\Delta(\mu_{\mathcal{P}}(p))[(d_k(i), d_\ell(j)), (d_k(i+1)-1, d_\ell(j+1)-1)]$$

contains exactly one letter in F. Now we show that if p is a robust state, the bidimensional picture $\lambda_M(\lambda_\Delta(\mu_{\mathcal{P}}(p)))$ is e-erasable. If $v := \lambda_M(\lambda_\Delta(\mu_{\mathcal{P}}(p)))$ is not e-erasable, then there must exist $m, n \ge 0$ such that $v_{m,n} = e$, $v_{m,n'} \ne e$ for some n' and $v_{m',n} \ne e$ for some m'. By construction, the letter $p' = (\mu_{\mathcal{P}}(p))_{m,n} = (q, k, \ell)$ is mapped to e either if $T_{p'} = \Delta$ or if $T_{p'} = M$. By the same reason, the letters $v_{m,n'} = (q', k, \ell')$ and $v_{m',n} = (q'', k', \ell)$ must be robust. Thus, there exist letters $a_{m''}$, $a_{n''} \in \Sigma_{\#}$ such that $\delta_{\mathcal{L}}(k, a_{m''}) \in F$ and $\delta_{\mathcal{L}}(\ell, a_{n''}) \in F$. Hence, it follows that $p' = (q, k, \ell)$ is robust, since the letter $(\mu_{\mathcal{P}}(p'))_{m'',n''}$ belongs to F, which is a contradiction. Then for each robust letter $p = (q, k, \ell)$, for each i with $0 \le i < h_k$ and for each j with $0 \le j < h_\ell$, write

$$(\rho_e(\lambda_M(w_p(i,j))))_{m,n} =: v_{p,i,j}(m,n)$$

where $(m,n) < \mathbf{s}_{p,i,j} := |\rho_e(\lambda_M(w_p(i,j)))|$. Note that the picture $\lambda_M(w_p(i,j))$ is e-erasable as a factor of the e-erasable picture $\lambda_M(\lambda_\Delta(\mu_\mathcal{P}(p)))$. Now we are ready to introduce a 2-dimensional morphism μ on a new alphabet Ξ and a coding $\nu' \colon \Xi \to Q$ such that $y = \nu'(\mu^\omega(\alpha))$ for a letter $\alpha \in \Xi$. The alphabet of new symbols is

$$\Xi = \{\alpha(p,i,j) \mid p = (q,k,\ell) \text{ is robust, } 0 \le i < h_k \text{ and } 0 \le j < h_\ell\}.$$

We define the bidimensional pictures $u_{p,i,j}(m,n)$ for each robust letter $p = (q,k,\ell) \in Q$, $(i,j) \in [\![0, h_k - 1]\!] \times [\![0, h_\ell - 1]\!]$ and $(m,n) \le \mathbf{s}_{p,i,j}$ as follows. If $v_{p,i,j}(m,n) = (q',k',\ell')$, then $u_{p,i,j}(m,n)$ is a picture of shape $(h_{k'}, h_{\ell'})$ such that

$$(u_{p,i,j}(m,n))_{i',j'} = \alpha(v_{p,i,j}(m,n), i', j')$$

for $(i',j') \in [\![0, h_{k'} - 1]\!] \times [\![0, h_{\ell'} - 1]\!]$. The image of $\alpha(p,i,j)$ by morphism $\mu \colon \Xi \to \mathcal{B}_2(\Xi)$ is defined as the word

$$\boxed{\begin{array}{cccc} u_{p,i,j}(0,0) & u_{p,i,j}(1,0) & \cdots & u_{p,i,j}(s_1-1,0) \\ u_{p,i,j}(0,1) & u_{p,i,j}(1,1) & \cdots & u_{p,i,j}(s_1-1,1) \\ \vdots & \vdots & \ddots & \vdots \\ u_{p,i,j}(0,s_2-1) & u_{p,i,j}(1,s_2-1) & \cdots & u_{p,i,j}(s_1-1,s_2-1) \end{array}}\,,$$

where $(s_1, s_2) = \mathbf{s}_{p,i,j}$. Note that the above concatenation of the pictures $u_{p,i,j}(m,n)$ is well defined. Since all letters occurring on a row of $w_p(i,j)$ are of the form (q',k',ℓ') where the third component ℓ' is fixed, it means that also the letters $v_{p,i,j}(m,n)$ and $v_{p,i,j}(m',n)$ occurring on the same row of $\rho_e(\lambda_M(w_p(i,j)))$ have the same third component ℓ'. Hence, $|u_{p,i,j}(m,n)|_{\widehat{1}} = |u_{p,i,j}(m',n)|_{\widehat{1}} = h_{\ell'}$ and the words $u_{p,i,j}(m,n)$ and $u_{p,i,j}(m',n)$ can be concatenated in the direction 1. The same holds for $u_{p,i,j}(m,n)$ and $u_{p,i,j}(m,n')$ in the direction 2. The coding $\nu' \colon \Xi \to Q$ is defined by

$$\nu'(\alpha(p,i,j)) = \rho(w_p(i,j))). \tag{4}$$

Note that by the definition of $w_p(i,j)$, there is only one letter belonging to F and the picture $\lambda(w_p(i,j))$ is e-erasable, since only one letter is different from e. Following the proof of [1, Theorem 7.7.4], we may prove by induction that

(5)
$$\nu' \circ \mu^n \left(\boxed{\begin{array}{cccc} \alpha(p,0,0) & \alpha(p,1,0) & \cdots & \alpha(p,h_k-1,0) \\ \alpha(p,0,1) & \alpha(p,1,1) & \cdots & \alpha(p,h_k-1,1) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(p,0,h_\ell-1) & \alpha(p,1,h_\ell-1) & \cdots & \alpha(p,h_k-1,h_\ell-1) \end{array}} \right) = \rho \circ \mu_\mathcal{P}^{n+1}(p)$$

for all robust letters $p = (q,k,\ell)$ and for all $n \ge 0$.

Since $\mu_\mathcal{P}$ is prolongable on p_0 and $x = \vartheta({\mu_\mathcal{P}}^\omega(p_0)$ is a 2-dimensional infinite word, p_0 must be a robust letter. Therefore, we have $(w_{p_0}(0,0))_{0,0} = v_{p_0,0,0}(0,0) = p_0$. Thus, $(u_{p_0,0,0}(0,0))_{0,0} = \alpha(p_0,0,0)$ and, consequently, the morphism μ is prolongable on $\alpha := \alpha(p_0,0,0)$. By (5), we have

$$\begin{aligned} \nu'(\mu^{n+1}(\alpha)) &= \begin{bmatrix} \nu'(\mu^n(u_{p_0,0,0}(0,0))) & U \\ V & W \end{bmatrix} \\ &= \begin{bmatrix} \rho(\mu_\mathcal{P}^{n+1}(p_0))) & U \\ V & W \end{bmatrix}, \end{aligned}$$

for all $n \ge 0$, where U, V and W are bidimensional pictures. Since $\rho(\mu_\mathcal{P}^{n+1}(p_0))$ tends to y as n tends to infinity, we have

$$\nu'(\mu^\omega(\alpha)) = \rho(\mu_\mathcal{P}^\omega(p_0)) = y.$$

Hence, defining the coding $\nu\colon \Xi \to \Gamma$ as $\nu = \tau \circ \nu'$ we obtain

$$\nu(\mu^\omega(\alpha)) = \tau(y) = x.$$

Example 42. Let us continue Example 38. Recall that the product automaton $\mathcal{P}$ is produced from the automaton $\mathcal{A}$ depicted in Figure 1 and the automaton $\mathcal{L}$ depicted in Figure 8. Note that the type of the state ℓ in $\mathcal{L}$ is $T_\ell = \Delta$ and all other states have type R_F. By Figure 9, we see that

$$\mu_{\mathcal{P}}(p,g,g) = \boxed{\begin{array}{ccc}(p,g,g) & (q,h,g) & (q,k,g)\\ (p,g,h) & (p,h,h) & (s,k,h)\\ (q,g,k) & (p,h,k) & (s,k,k)\end{array}}$$

and

$$\mu_{\mathcal{P}}(q,h,g) = \boxed{\begin{array}{ccc}(q,\ell,g) & (p,h,g) & (q,k,g)\\ (p,\ell,h) & (s,h,h) & (q,k,h)\\ (p,\ell,k) & (q,h,k) & (s,k,k)\end{array}}\,.$$

Since h_ℓ is the number of letters $a_n \in \Sigma_\#$ such that $\delta_{\mathcal{L}}(\ell, a_n) \in F_{\mathcal{L}}$, we notice that $h_g = 3$ and $h_h = 2$. By Figure 10, we have $\rho_e(\lambda_\Delta(\mu_{\mathcal{P}}(p,g,g))) = \rho_e(\mu_{\mathcal{P}}(p,g,g)) = \mu_{\mathcal{P}}(p,g,g)$ and

$$\rho_e(\lambda_\Delta(\mu_{\mathcal{P}}(q,h,g))) = \rho_e\left(\boxed{\begin{array}{ccc}e & (p,h,g) & (q,k,g)\\ e & (s,h,h) & (q,k,h)\\ e & (q,h,k) & (s,k,k)\end{array}}\right) = \boxed{\begin{array}{cc}(p,h,g) & (q,k,g)\\ (s,h,h) & (q,k,h)\\ (q,h,k) & (s,k,k)\end{array}}\,.$$

Since all letters in $\lambda_\Delta(\mu_{\mathcal{P}}(p,g,g)) = \mu_{\mathcal{P}}(p,g,g)$ belong to F, the picture $w_{(p,g,g)}(i,j)$ is a square of size 1 for $(i,j) \in [\![0, h_g - 1]\!] \times [\![0, h_g - 1]\!]$. Consequently,

$$\mathbf{s}_{(p,g,g),i,j} = |\rho_e(\lambda_M(w_{(p,g,g)}(i,j)))| = (1,1)$$

and

$$v_{(p,g,g),i,j}(0,0) = w_{(p,g,g)}(i,j) = (\mu_P(p,g,g))_{i,j}$$

for $(i,j) \in [\![0,2]\!] \times [\![0,2]\!]$. Especially, we have $v_{(p,g,g),0,0}(0,0) = (p,g,g)$ and $v_{(p,g,g),1,0}(0,0) = (q,h,g)$. Hence, $u_{(p,g,g),0,0}(0,0)$ is a picture of shape $(h_g,h_g) = (3,3)$ such that

$$(u_{(p,g,g),0,0}(0,0))_{i',j'} = \alpha(v_{(p,g,g),0,0}(0,0),i',j') = \alpha((p,g,g),i',j')$$

for $(i',j') \in [\![0,2]\!] \times [\![0,2]\!]$ and the image $\mu(\alpha((p,g,g),0,0)) = u_{(p,g,g),0,0}(0,0)$ is

$$\boxed{\begin{array}{ccc}\alpha((p,g,g),0,0) & \alpha((p,g,g),1,0) & \alpha((p,g,g),2,0)\\ \alpha((p,g,g),0,1) & \alpha((p,g,g),1,1) & \alpha((p,g,g),2,1)\\ \alpha((p,g,g),0,2) & \alpha((p,g,g),1,2) & \alpha((p,g,g),2,2)\end{array}}\,.$$

Similarly, $|u_{(p,g,g),1,0}(0,0)| = (h_h,h_g) = (2,3)$ and

$$(u_{(p,g,g),1,0}(0,0))_{i',j'} = \alpha(v_{(p,g,g),1,0)}(0,0),i',j') = \alpha((q,h,g),i',j')$$

for $(i',j') \in [\![0,1]\!] \times [\![0,2]\!]$. Thus, the image $\mu(\alpha((p,g,g),1,0)) = u_{(p,g,g),1,0}(0,0)$ is

$$\boxed{\begin{array}{cc}\alpha((q,h,g),0,0) & \alpha((q,h,g),1,0)\\ \alpha((q,h,g),0,1) & \alpha((q,h,g),1,1)\\ \alpha((q,h,g),0,2) & \alpha((q,h,g),1,2)\end{array}}\,.$$

Next we apply the coding ν to the images above. Note that

$$\begin{array}{rclrcl}
w_{(q,h,g)}(0,0) &=& \boxed{e \quad (p,h,g)}\,, & w_{(q,h,g)}(1,0) &=& \boxed{(q,k,g)}\,,\\
w_{(q,h,g)}(0,1) &=& \boxed{e \quad (s,h,h)}\,, & w_{(q,h,g)}(1,1) &=& \boxed{(q,k,h)}\,,\\
w_{(q,h,g)}(0,2) &=& \boxed{e \quad (q,h,k)}\,, & w_{(q,h,g)}(1,2) &=& \boxed{(s,k,h)}\,.
\end{array}$$

Hence, by (4), we have $\nu'(\mu(\alpha((p,g,g),0,0))) = \mu_{\mathcal{P}}(p,g,g)$ and

$$\nu'(\mu(\alpha((p,g,g),1,0))) = \boxed{\begin{matrix}(p,h,g) & (q,k,g)\\ (s,h,h) & (q,k,h)\\ (q,h,k) & (s,k,k)\end{matrix}}.$$

Since $\nu = \tau \circ \nu'$, the infinite word $\nu(\mu^\omega(\alpha((p,g,g),0,0)))$ begins with

$$\nu\left(\mu(\alpha((p,g,g),0,0)) \odot^1 \mu(\alpha((p,g,g),1,0))\right) = \boxed{\begin{matrix}p & q & q & p & q\\ p & p & s & s & q\\ q & p & s & q & s\end{matrix}},$$

which is exactly the left upper corner of the infinite word depicted in Figure 2.

Finally, we have to show that $w = \mu^\omega(\alpha)$ is shape-symmetric, that is for all $m,n \geq 0$, if $|\mu(w_{m,n})| = (s,t)$ then $|\mu(w_{n,m})| = (t,s)$. First, observe that if $p = (q,k,\ell)$ is a robust letter of Q, $0 \leq i < h_k$ and $0 \leq j < h_\ell$, then the shape of $\mu(\alpha(p,i,j))$ does not depend on q. More precisely, we have

$$|\mu(\alpha(p,i,j))| = \left(\sum_{m=0}^{s_1-1} |u_{p,i,j}(m,0)|_1, \sum_{n=0}^{s_2-1} |u_{p,i,j}(0,n)|_2\right), \tag{6}$$

where $(s_1,s_2) = \mathbf{s}_{p,i,j}$ does not depend on q, the component $|u_{p,i,j}(m,0)|_1$ does not depend on q, ℓ and j and, similarly, $|u_{p,i,j}(0,m)|_2$ does not depend on q, k and i. Moreover, for all $d \geq 0$, we have $|\mu^d(\alpha)| = (t_d,t_d)$ for some integer $t_d \geq 0$, since $\alpha = \alpha(p_0,0,0)$ where the second and the third component of $p_0 = (q_0,l_0,l_0)$ are equal. Hence, it suffices to show for all $m,n \geq 0$ that if $w_{m,n} = \alpha((q,k,\ell),i,j)$ then $w_{n,m} = \alpha((q',\ell,k),j,i)$ for some q' in $Q_{\mathcal{A}}$. We prove this by induction on the power d of μ. Assume that for all $m,n \in [\![0,t_d-1]\!]$, if $(\mu^d(\alpha))_{m,n} = \alpha((q,k,\ell),i,j)$ then $(\mu^d(\alpha))_{n,m} = \alpha((q',\ell,k),j,i)$ for some $q' \in Q_{\mathcal{A}}$. For $d = 0$, the assumptions are clearly satisfied. Consider now the letter

$$w_{m,n} = (\mu^{d+1}(\alpha))_{m,n} =: \alpha((q,k,\ell),i,j),$$

where $m,n \in [\![0,t_{d+1}-1]\!]$ and m or n belongs to $[\![t_d,t_{d+1}-1]\!]$. There exist unique $m',n' \in [\![0,t_d-1]\!]$ such that $w_{m,n}$ is generated by applying μ to

$$w_{m',n'} = (\mu^d(\alpha))_{m',n'} =: \alpha((q',k',\ell'),i',j').$$

By definition of μ, there exists a unique pair $(m'',n'') < \mathbf{s}_{(q',k',l'),i',j'}$ such that

$$(u_{(q',k',\ell'),i',j'}(m'',n''))_{i,j} = \alpha(v_{(q',k',\ell'),i',j'}(m'',n''),i,j) = w_{m,n}.$$

By induction hypothesis, we can write

$$w_{n',m'} = (\mu^d(\alpha))_{n',m'} = \alpha((q'',\ell',k'),j',i'),$$

where $q'' \in Q_{\mathcal{A}}$ and by (6) we have

$$(|\mu(w_{n',m'})|_1, |\mu(w_{n',m'})|_2) = (|\mu(w_{m',n'})|_2, |\mu(w_{m',n'})|_1).$$

Therefore $w_{n,m}$ must be generated by applying μ to $w_{n',m'}$. Moreover

$$\begin{aligned}(|u_{(q'',\ell',k'),j',i'}(n'',m'')|_1, &|u_{(q'',\ell',k'),j',i'}(n'',m'')|_2)\\ &= (|u_{(q',k',\ell'),i',j'}(m'',n'')|_2, |u_{(q',k',\ell'),i',j'}(m'',n'')|_1).\end{aligned}$$

Thus, we conclude that

$$(u_{(q'',\ell',k'),j',i'}(n'',m''))_{j,i} = \alpha(v_{(q'',\ell',k'),j',i'}(n'',m''),j,i) = w_{n,m}.$$

Therefore we get that $v_{(q'',\ell',k'),j',i'}(n'',m'') = (q''',\ell,k)$ for some $q''' \in Q_{\mathcal{A}}$. Hence,

$$w_{n,m} = (\mu^{d+1}(\alpha))_{n,m} = \alpha((q''',\ell,k),j,i)$$

and the result follows.

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Institute of Mathematics, University of Liège, Grande Traverse 12 (B 37), B-4000 Liège, Belgium

E-mail address: {echarlier,T.Karki,M.Rigo}@ulg.ac.be