A DECISION PROBLEM FOR ULTIMATELY PERIODIC SETS IN NON-STANDARD NUMERATION SYSTEMS

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Definition 1. A *numeration system* is given by a (strictly) increasing sequence $U = (U_i)_{i \geq 0}$ of integers such that $U_0 = 1$ and $C_U := \sup_{i \geq 0} \lceil U_{i+1}/U_i \rceil$ is finite. Let $A_U = \{0, \ldots, C_U - 1\}$. The *greedy U-representation* of a positive integer n is the unique finite word $\mathrm{rep}_U(n) = w_\ell \cdots w_0$ over A_U satisfying

$$n = \sum_{i=0}^{\ell} w_i \, U_i, \;\; w_\ell \neq 0 \text{ and } \sum_{i=0}^{t} w_i \, U_i < U_{t+1}, \;\; \forall t = 0, \ldots, \ell.$$

We set $\mathrm{rep}_U(0)$ to be the empty word ε. A set $X \subseteq \mathbb{N}$ of integers is *U-recognizable* if the language $\mathrm{rep}_U(X)$ over A_U is regular (i.e., accepted by a finite automaton). If $x = x_\ell \cdots x_0$ is a word over a finite alphabet of integers, then the *U-numerical value* of x is

$$\mathrm{val}_U(x) = \sum_{i=0}^{\ell} x_i \, U_i.$$

Definition 2. A numeration system $U = (U_i)_{i \geq 0}$ is said to be *linear*, if the sequence U satisfies a homogenous linear recurrence relation. For all $i \geq 0$, we have

$$U_{i+k} = a_1 U_{i+k-1} + \cdots + a_k U_i \tag{1}$$

for some $k \geq 1$, $a_1, \ldots, a_k \in \mathbb{Z}$ and $a_k \neq 0$.

We address the following decidability question.

Problem 1. Given a linear numeration system U and a set $X \subseteq \mathbb{N}$ such that $\mathrm{rep}_U(X)$ is recognized by a (deterministic) finite automaton. Is it decidable whether or not X is ultimately periodic, i.e., whether or not X is a finite union of arithmetic progressions ?

J. Honkala showed in [1] that Problem 1 turns out to be decidable for the usual integer base $b \geq 2$ numeration system defined by $U_n = b \, U_{n-1}$ for $n \geq 1$.

In this work, we give a decision procedure for Problem 1 whenever U is a linear numeration system such that $\mathbb{N}$ is U-recognizable and satisfying a relation like (1) with $a_k = \pm 1$ (the main reason for this assumption is that 1 and -1 are the only two integers invertible modulo n for all $n \geq 2$).

Theorem 3. *Let $U = (U_i)_{i \geq 0}$ be a linear numeration system such that $\mathbb{N}$ is U-recognizable and satisfying a recurrence relation of order k of the kind* (1) *with $a_k = \pm 1$ and $\lim_{i \to +\infty} U_{i+1} - U_i = +\infty$. It is decidable whether or not a U-recognizable set is ultimately periodic.*

In a second part, we consider the same decision problem but restated in the framework of abstract numeration systems [2]. We apply successfully the same kind of techniques to a large class of abstract numeration systems.

Definition 4. [2] An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where L is an infinite regular language L over a totally ordered alphabet Σ. The genealogical order (words are ordered by increasing length and for words of same length, one uses the lexicographical ordering induced by the total ordering $<$ on the alphabet Σ) gives a one-to-one correspondence denoted rep_S between $\mathbb{N}$ and L. In particular, 0 is represented by the first word in L. The reciprocal map associating a word $w \in L$ to its index in the genealogically ordered language L is denoted val_S (the first word in L having index 0). A set $X \subseteq \mathbb{N}$ of integers is *S-recognizable* if the language $\mathrm{rep}_S(X)$ over Σ is regular (i.e., accepted by a finite automaton).

We denote by $\mathcal{M}_L = (Q_L, q_{0,L}, \Sigma, \delta_L, F_L)$ the minimal automaton of L. The transition function $\delta_L : Q_L \times \Sigma \to Q_L$ is extended on $Q_L \times \Sigma^*$ and we denote by $\mathbf{u}_j(q)$ (resp. $\mathbf{v}_j(q)$) the number of words of length j (resp. $\leq j$) accepted from $q \in Q_L$ in $\mathcal{M}_L$.

We consider the following decidability question analogous to Problem 1.

Problem 2. Given an abstract numeration system S and a set $X \subseteq \mathbb{N}$ such that $\mathrm{rep}_S(X)$ is recognized by a (deterministic) finite automaton. Is it decidable whether or not X is ultimately periodic, i.e., whether or not X is a finite union of arithmetic progressions ?

We give a decision procedure for Problem 2 whenever S satisfies some extra hypothesis.

Theorem 5. *Let $S = (L, \Sigma, <)$ be an abstract numeration system such that for all states q of the trim minimal automaton $\mathcal{M}_L = (Q_L, q_{0,L}, \Sigma, \delta_L, F_L)$ of L*

$$\lim_{j\to\infty} \mathbf{u}_j(q) = +\infty$$

and $\mathbf{u}_j(q_{0,L}) > 0$ for all $j \geq 0$. Assume moreover that $(\mathbf{v}_i(q_{0,L}))_{i\geq 0}$ satisfies a linear recurrence relation of the form (1) with $a_k = \pm 1$. It is decidable whether or not a S-recognizable set is ultimately periodic.

References

[1] J. Honkala, A decision method for the recognizability of sets defined by number systems, *Theoret. Inform. Appl.* **20** (1986), 395–403.

[2] P.B.A. Lecomte, M. Rigo, Numeration systems on a regular language, *Theory Comput. Syst.* **34** (2001), 27–44.

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