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# Lifting of valid inequalities revisited

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## Outline

1. Introduction on lifting

2. Basic theory

3. Superadditive lifting

4. Conclusion

## Introduction

## Valid inequalities from subsets and supersets

$X = \{z \in \mathbb{R}_+^{n_1} \times \mathbb{Z}_+^{n_2} : Az \leq b\}$ $\qquad$ $Y = X \cap \{z : Cz = e\}$

- One can find facets of $Y$ from facets of $X$. If $Cz \leq e$ is true for all $z \in X$, $\text{conv}(Y) = \text{conv}(X) \cap \{z : Cz = e\}$.

- Finding facets of $X$ from facets of $Y$: The **lifting** problem.

  $\pi_1 z \leq \pi_0$ valid for $Y$. Does there exist $\pi_2$ such that

  $$\pi_1 z + \pi_2(e - Cz) \leq \pi_0$$

  is valid for $X$?

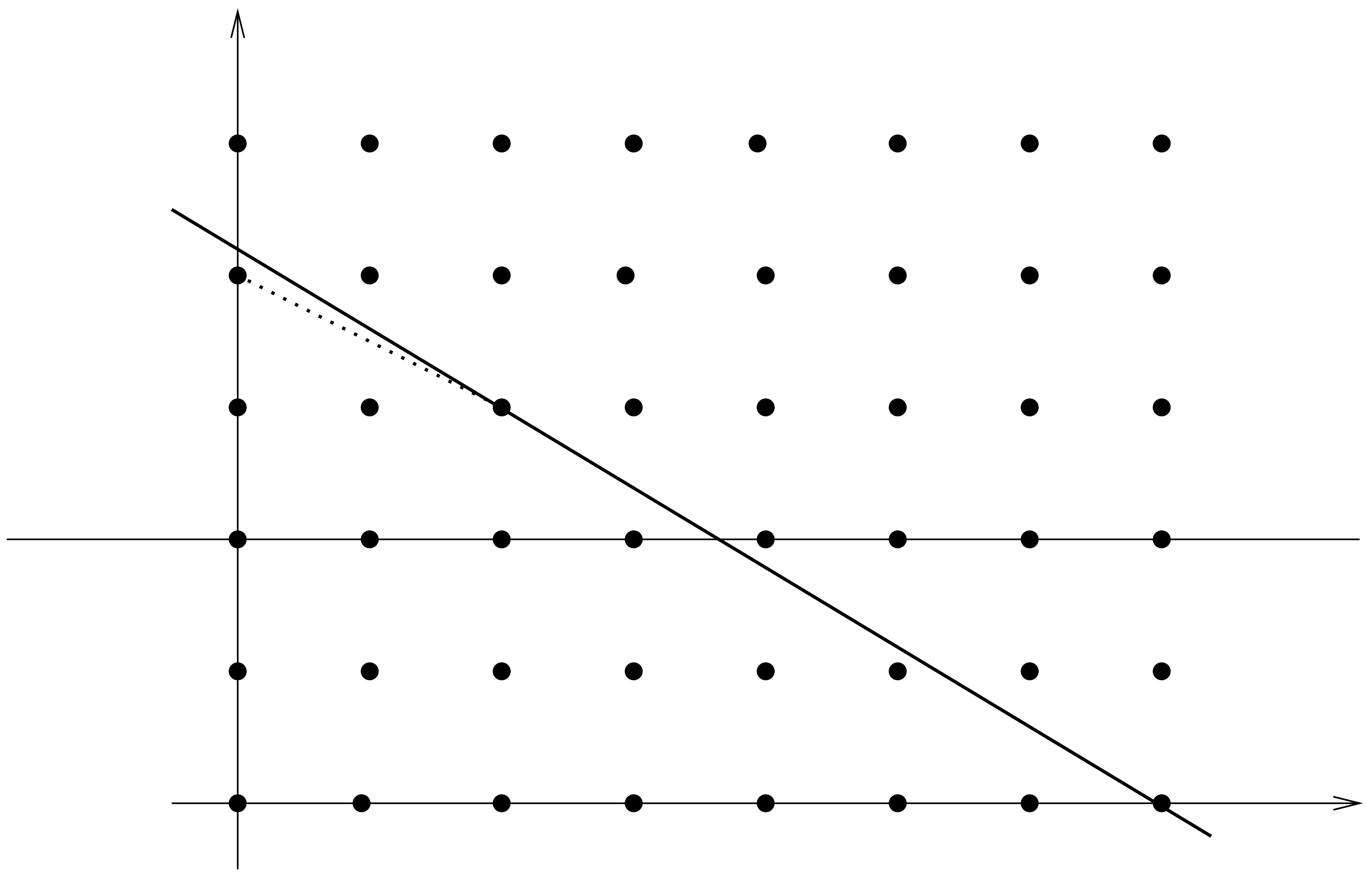
## Always better when the smaller set is a face

- If $Cz \leq e$ for all $z \in X$, it is always possible to find $\pi_2$ such that

$$\pi_1 z + \pi_2(e - Cz) \leq \pi_0 \text{ is valid for } X.$$

- If $Cz \not\leq e$, it can happen that no multipliers $\pi^2$ exist.

  *Ex:* $X = \{y \in \mathbb{Z}^2_+ : 3y_1 + 5y_2 \leq 21, y_2 \leq 4\}$ and $Y = X \cap \{y_2 = 2\}$.
  Valid inequality for $Y$: $y_1 \leq 3$.
  $\pi_2$? such that $y_1 + \pi_2(y_2 - 2) \leq 3$ is valid for $X$?
  (7,0) is valid $\Rightarrow \pi_2 \geq 2$
  (0,4) is valid $\Rightarrow \pi_2 \leq 3/2$



## Lifting: Basic Theory

We consider a mixed-integer set of the form

$$Z(b) = \begin{array}{c} A^1 z^1 + A^2 z^2 \leq b + s \\ z^1 \in X^1, z^2 \in X^2, s \in \mathbb{R}^m_+. \end{array}$$

- Hypothesis: $0 \in X^1, 0 \in X^2$.
- General idea: Fix $z^2 = 0$, find a valid inequality, lift it to a valid inequality for $Z(b)$.
- Not restrictive to fix $z^2 = 0$, by choosing the right representation.
  *Example:* For fixing $z^2 = 2$, write $\hat{z}^2 = z^2 - 2$.
  For fixing $x = ay$, write $\hat{z} = x - ay$.

## The approach

1. Fix $z^2 = 0$.

2. Find the valid inequality $\pi^1 z^1 \leq \pi_0 + \nu s$ for $Z^1(b)$.

3. Lift the variables $z^2$ and find $\pi^2$ such that

$$\pi^1 z^1 + \pi^2 z^2 \leq \pi_0 + \nu s$$

   is valid or determine that no such $\pi^2$ exists.

## The lifting function

**Definition 1** *The lifting function* $\phi^1 : \mathbb{R}^m \rightarrow \mathbb{R}^1$ *is*

$$\phi^1(u) = \min\{\pi_0+\nu s-\pi^1 z^1 : (z^1, s) \in Z^1(b-u)\}.$$

When the variables sum up to $u$ in the constraints, what happens in the valid inequality?

**Definition 2**

$$\Pi^2 = \{\pi : \pi t \leq \phi^1(A^2 t) \text{ for all } t \in X^2\}.$$

**Proposition 3**

$\pi^1 z^1+\pi^2 z^2 \leq \pi_0+\nu s$ *is valid for* $Z(b)$ *iff* $\pi_2 \in \Pi^2$.

## Lifting: A first example

Consider the set

$X = \{(x,s) \in \{0,1\}^4 \times \mathbb{R}_+ : 2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 6+$

If we fix $x_1 = x_2 = x_3 = 0$, we obtain

$$Y = \{(x,s) \in \{0,1\} \times \mathbb{R}_+ : 5x_4 \leq 6 + s\}.$$

MIR procedure: valid inequality for $Y$: $4x_4 \leq 4 + s$.

The lifting function:

$$\phi(u) = \min\{4 - 4x_4 + s : 5x_4 \leq 6 + s - u\}$$

Ex: $3x_3 + 4x_4 \leq 4 + s$ $\qquad$ $x_1 + 2x_2 + 4x_4 \leq 4 + s$

valid for $X$.

## How to compute $\pi^2$ in general?

In some cases, computing $\pi^2$ is not obvious. However, we have this result.

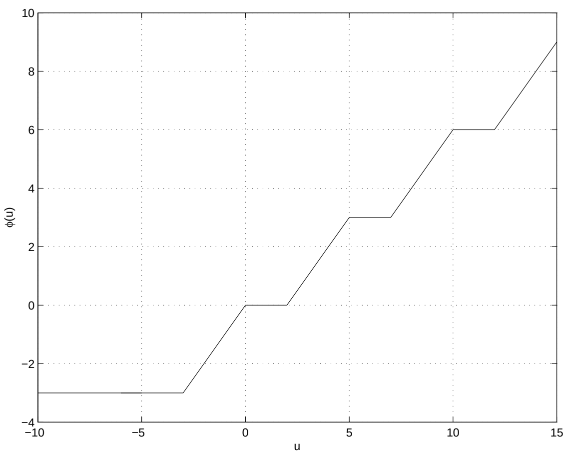
**Proposition 4** *If $X^1$ and $X^2$ are bounded mixed-integer sets, $\Pi^2$ is a polyhedron.*

Usually, $\Pi^2$ is described by inequalities found at "singular points" of $\phi$ and "discontinuity points" of the domain.

**Second example:**

$$5y_1 + 5y_2 + 5y_3 + x_4 + 2y_4 \leq 12 + s$$
$$y_4 \leq x_4 \leq 3y_4$$
$$y_1, y_2, y_3 \in \{0,1\}, y_4 \in \{0,1,2\}, x_4 \in \mathbb{R}_+$$

Fix $x_4 = y_4 = 0$, valid inequality: $3y_1 + 3y_2 + 3y_3 \leq 6 + s$.




$$\lambda + \mu \leq \phi(3) = 1$$
$$3\lambda + \mu \leq \phi(5) = 3$$
$$\lambda + 2\mu \leq \phi(5) = 3$$
$$3\lambda + 2\mu \leq \phi(7) = 3$$
$$6\lambda + 2\mu \leq \phi(10) = 6$$

## An issue: computing the new lifting function

**Proposition 5**

$$\phi^2(u) = \min_{t \in X^2} [\phi^1(u + A^2 t) - \pi^2 t].$$

When the lifting function is superadditive, the situation simplifies.

**Definition 6** *A function* $F : \mathbb{R}^m \to \mathbb{R}$ *is* super-additive *on* $D \subseteq \mathbb{R}^m$ *if* $F(0) = 0$ *and*

$$F(u) + F(v) \leq F(u + v).$$

**Proposition 7** *If* $\phi^1$ *is superadditive,* $\phi^2 = \phi^1$*.*

## Sequence independent lifting

Even if $\phi^1$ is not superadditive, a function $\widehat{\phi} \leq \phi^1$ that is superadditive and close to $\phi^1$ can be useful.

**Proposition 8** *If $\widehat{\phi} \leq \phi^1$ and $\widehat{\phi}$ is superadditive and nondecreasing, $\widehat{\phi}$ can be used for lifting, and $\phi^1 \geq \phi^2 \geq \widehat{\phi}$.*

In that case, the ordering of the variables lifted is irrelevant which is not true in the general case.

## Conclusion

- By fixing all variables to 0, we avoid the use of 2 lifting functions.
- No loss of generality by this restriction.
- Keeping a continuous variable $s$ simplifies the theory ($\phi$ always exists and is continuous).
- Computing of superadditive lower bounds is an important issue.
- No real clue on how to do it efficiently.