Outline

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- Four Extended Formulations of Corner Polyhedra
  - Disaggregated
  - Aggregated
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- Computational Results and Conclusion

Introduction: The Group Problem

$$
\begin{array}{ll}
\max & c^T x \\
\text{s.t.} & Ax = b \\
& x \in \mathbb{Z}^n_+
\end{array}
$$

Vertex with basic variables x_B and non basic x_N,

$$
\begin{aligned}
\max\ & \bar{c}_N^T x_N \\
\text{s.t.}\ & x_B + \bar{A}_N x_N = \bar{b} \\
& x \in \mathbb{Z}^n_+
\end{aligned} \tag{1}
$$

Relax nonnegativity on x_B in (1). New problem:

$$
\begin{aligned}
& \bar{A}_N x_N \equiv \bar{b} \pmod 1 \\
& x_N \in \mathbb{Z}^{|N|}_+ .
\end{aligned} \tag{2}
$$

Convex hull of (2) is called a **Corner Polyhedron**.

Example:

$$\begin{array}{rrrrrrl}
\max & 2x_1 & +3x_2 & & & & \\
\text{s.t.} & 5x_1 & +9x_2 & +s_1 & & & = 35 \\
 & 11x_1 & +4x_2 & & +s_2 & & = 45 \\
 & x_1 + & x_2 & & & +s_3 & = 5 \\
 & x_1, & x_2, & s_1, & s_2, & s_3 & \in \mathbb{Z}_+.
\end{array}$$

The optimal – fractional – tableau is

$$\begin{array}{rrrrrl}
\max & & -\frac{1}{4}s_1 & & -\frac{3}{4}s_3 & \\
\text{s.t.} & x_1 & -\frac{1}{4}s_1 & & +\frac{9}{4}s_3 & = \frac{10}{4} \\
 & x_2 & +\frac{1}{4}s_1 & & -\frac{5}{4}s_3 & = \frac{10}{4} \\
 & & \frac{7}{4}s_1 & +s_2 & -\frac{79}{4}s_3 & = \frac{15}{2} \\
 & x_1, x_2, & s_1, & s_2, & s_3 & \in \mathbb{Z}_+.
\end{array}$$

Corner Polyhedron is $\{s_1, s_3 \in \mathbb{Z}_+ : 3s_1 + s_3 \equiv 2 \pmod 4\}$.

The work of Gomory

- For a **facet** description, it is enough to study the **master problem**

$$x_1 + 2x_2 + 3x_3 + \cdots + (d-1)x_{d-1} \equiv b \pmod{d}.$$

- Gomory gives a complete description of the facets as extreme rays of a polyhedron.
- Computionally limited because of the sizes of the groups

First reformulation based on irreducibles

Definition

A vector $y \in \mathbb{Z}_+^n$ is an **irreducible solution** of $Ax = b$ if every nonzero solution $z \in \mathbb{Z}_+^n$ of $Ax = b$ is such that $z \not\leq y$.

- **Inhomogeneous** irreducibles: solutions of $Ax = b \to C$
- **Homogeneous** irreducibles: solutions of $Ax = 0 \to D$

Proposition

Every solution $y \in \mathbb{Z}_+^n$ of $Ax = b$ can be written

$$
\begin{aligned}
& y = C\lambda + D\mu \\
& 1 \cdot \lambda = 1 \\
& \lambda \in \mathbb{Z}_+^s, \mu \in \mathbb{Z}_+^t
\end{aligned}
$$

A first reformulation of a group problem

Basic: single-row group problem

$$Y(f) = \{x \in \mathbb{Z}^n_+ : bx \equiv f \pmod d\}. \qquad (3)$$

Computation of 2 matrices:
- C: inhomogeneous irreducible solutions
- D: homogeneous irreducible solutions

Proposition

$$Y(f) = \{x \in \mathbb{R}^n_+ : x = C\lambda + D\mu, 1 \cdot \lambda = 1, \lambda \in \mathbb{Z}^s_+, \mu \in \mathbb{Z}^t_+\}$$

which leads to a valid extended formulation of (3).

Example:

$$Y(2) = \{3x_1 + 3x_2 + x_3 \equiv 2 \pmod 4\}$$

Irreducibles:

$$C = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Valid reformulation:

$$\begin{aligned} Y(2) = \{x \in \mathbb{Z}^3_+ : \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_8 \end{pmatrix} \\ \lambda_1 + \cdots + \lambda_4 = 1 \\ \lambda_1, \ldots, \lambda_4 \in \{0, 1\}, \; \mu_1, \ldots, \mu_8 \in \mathbb{Z}_+ \quad \}. \end{aligned}$$

The convex hull property is satisfied

Notation: P_Y^1 is the polyhedron obtained by relaxing the integrality requirements in the reformulation.

Proposition

$P_Y^1 = \text{conv}(Y(f))$.

Proof: The extreme points are inhomogeneous irreducible solutions and the extreme rays are parallel to some homogeneous irreducible solutions. □

Conclusion: Reformulating is, in LP terms, as strong as adding the facets.

Drawback: There can be many irreducibles.

Second reformulation: aggregate variables

Many irreducibles come from variables with identical coefficients.

Idea: Aggregate variables with the same coefficient.

$$w_\alpha = \sum_{j:b_j \equiv \alpha} x_j$$

$W(f) = \{w \in \mathbb{Z}_+^d : \sum_{\alpha=0}^{d-1} \alpha w_\alpha \equiv f \pmod d\}$

- $\tilde{C}$: inhomogeneous irreducible solutions for $W(f)$
- $\tilde{D}$: homogeneous irreducible solutions for $W(f)$

Proposition

$$Y(f) = \{x \in \mathbb{Z}_+^n : w = \tilde{C}\lambda + \tilde{D}\mu,\ 1\lambda = 1,\ \lambda \in \mathbb{Z}_+^{\tilde{s}},\ \mu \in \mathbb{Z}_+^{\tilde{t}},\ w_\alpha = \sum_{j:b_j \equiv \alpha} x_j,\ w \in \mathbb{Z}_+^d\}.$$

which leads to a valid extended formulation called **aggregated formulation**.

Example:

$$Y(2) = \{x \in \mathbb{Z}^3_+ : 3x_1 + 3x_2 + x_3 \equiv 2 \pmod 4\}$$

Aggregation: $w = x_1 + x_2$

$$W(2) = \{w, x_3 \in \mathbb{Z}_+ : 3w + x_3 \equiv 2 \pmod 4\}$$

Irreducibles:

$$\tilde{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

Valid reformulation:

$$Y(2) = \{x \in \mathbb{Z}^3_+ : x_1 + x_2 = w$$

$$\begin{pmatrix} w \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix} \Bigg($$

$$\lambda_1 + \lambda_2 = 1,\ \lambda \in \mathbb{Z}^2_+,\ \mu \in \mathbb{Z}^3_+,\ w \in \mathbb{Z}_+\}.$$

Properties

Notation: P_Y^2 is the polyhedron obtained from relaxing the integrality constraints in the aggregated formulation.

Proposition

$P_Y^2 = \text{conv}(Y(f))$

- Reduce the number of irreducibles
- A much more compact reformulation with the same LP bound

Advanced aggregation

- Idea: Aggregate variables with different coefficients.
  Example: $w = x_1 + 3x_3$
- Reformulation with irreducibles valid
  Convex hull property does not hold anymore.
- Interesting to study the polyhedron coming from the constraints

$$x_1 + hx_2 = \sum_i c_i \lambda_i + \sum_j d_j \mu_j$$

$$\sum_i \lambda_i = 1$$

Path reformulation

- Corresponds to the path structure of the group problem
- One node for each group element
  Arcs $(\alpha, \alpha + b_j \pmod{d})$ for each α and each variable j
- A solution is a path from 0 to f

An interesting question: how to compute the irreducibles

- Use of Buchberger-type algorithm or lexicographic enumeration: exponential methods not suited for use in an iterative algorithm
- Possible to precompute the irreducibles once and store them in a table
  $\rightarrow$ Reading the table is fast
- One can use group automorphisms to reduce the size of the table

Sizes of the reformulations

Example:

$$3x_1 + 3x_2 + 3x_3 + 6x_4 + 5x_5 + 10x_6 + 7x_7 \equiv 1 \pmod{11}$$

Formulation	Homogeneous Irreducibles	Inhomogeneous Irreducibles	Variables
Disaggregated	378	76	454
Aggregated	54	26	80
Advanced aggregation	13	8	21
Path			77

The group reformulation in a primal setting

Starting point: a primal integer tableau

$$\begin{array}{ll} \max & \bar{c}_N^T x_N \\ \text{s.t.} & x_B + \bar{A}_N x_N = \bar{b} \\ & x \in \mathbb{Z}_+^n. \end{array}$$

Method

- Select a row
- Consider it as a modulo row and generate a group reformulation
- Recover a primal feasible integer tableau
- Possibly find augmentation vectors in the new variables

The use of aggregated reformulation is not easy!

Group reformulation also provides augmentation vectors

Integral Basis Method (Haus, Köppe and Weismantel)

Problem Name: lseu (Optimal value : 1120)					
Objective	Row	Modulus	Result	New Obj	GAP close
1660	R123	mod 10	augmentation	1472	35 %
1472	R123	mod 10	augmentation	1303	66 %
No augmentation found by group!					

Problem Name: p0282 (Optimal value : 258411)					
Objective	Row	Modulus	Result	New Obj	GAP clo
366777	R1026	mod 5	augmentation	329640	34 %
329640	R1026	mod 5	augmentation	325655	38 %
325655	R1056	mod 5	augmentation	322538	41 %
No augmentation found by group!					

Conclusion

- Operation of reformulating is as strong as adding facets.
  In both cases, the master problem is enough.
- Possibility of finding augmenting vectors in a reformulation.
  Aggregating makes the task more complicated.
- Hope: primal-dual algorithm with fractional pivots.