Polyhedral properties for the intersection of two knapsacks

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Outline

- Introduction
  Forbidden minors for the intersection of two knapsacks

- Incomplete Set Inequalities

- Strength
  Compared to the intersection of the convex hulls of the single knapsacks

- The mixed case

Introduction

- Generation of valid inequalities: focus on one constraint.
  Example: Gomory cuts, MIR, cover inequalities.

- No general work done on several constraints at a time.
  Commercial softwares only generate inequalities from one constraint.

- Missing knowledge: when is it enough to consider one and when do we need to consider more constraints?

- Starting point: The intersection of two $\{0, 1\}-$knapsacks.

Combinatorial valid inequalities for the intersection of two knapsacks

Problem:

$$X_1 = \{x \in \{0,1\}^n : \sum_{i \in N} a_i x_i \leq a_0\}$$
$$X_2 = \{x \in \{0,1\}^n : \sum_{i \in N} b_i x_i \leq b_0\}$$
$$X = \{x \in \{0,1\}^n : x \in X_1 \cap X_2\}$$

Central question: How to derive valid inequalities of the type

$$\sum_{i \in C} x_i \leq |C| - 1 \tag{1}$$

for X?

First Case: inequality valid for one single knapsack

Example:

$$X_1 = \{x \in \{0,1\}^3 : 2x_1 + 3x_2 + 4x_3 \leq 4\}$$
$$X_2 = \{x \in \{0,1\}^3 : 3x_1 + 5x_2 + 6x_3 \leq 8\}.$$

$$x_1 + x_2 \leq 1$$

valid for X_1 and hence for $X = X_1 \cap X_2$.

Observation 1 *If $a_i, b_i \geq 0$ and (1) is valid for X, then (1) is either valid for X_1 or for X_2.*

Canonical form for an intersection

In the following: both positive and negative signs are present!

Canonical form:

$$X_1 = \{x \in \{0,1\}^n : \sum_{i \in N_+} a_i x_i + \sum_{i \in N_-} a_i x_i \leq a_0\}$$

$$X_2 = \{x \in \{0,1\}^n : \sum_{i \in N_+} b_i x_i + \sum_{i \in N_-} b_i x_i \leq b_0\}$$

$$X = \{x \in \{0,1\}^n : x \in X_1 \cap X_2\},$$

with

$N_+ = \{i \in N : a_i \geq 0, b_i \geq 0\}$ and $N_- = \{i \in N : a_i \geq 0, b_i < 0\}$.

Second Case: Valid for an aggregation of the constraints

Example:

$$X_1 = \{x \in \{0,1\}^4 : 7x_1 + 10x_2 + 13x_3 + 12x_4 \leq 25\} \tag{2}$$
$$X_2 = \{x \in \{0,1\}^4 : x_1 + 2x_2 - 2x_3 - x_4 \leq 1\}. \tag{3}$$

$$x_1 + x_2 \leq 1$$

valid for X but neither valid for X_1 nor for X_2.

Derivation: (2)+5(3) yields

$$12x_1 + 20x_2 + 3x_3 + 7x_4 \leq 30.$$

$\{1,2\}$ is a cover!

Simple second constraint =

Derivation always possible by aggregation

Theorem 1 *Let*

$$X = \{x \in \{0,1\}^n : \sum_{i \in N_+} a_i x_i + \sum_{j \in N_-} a_j x_j \leq a_0 \tag{4}$$

$$\sum_{i \in N_+} x_i - \sum_{j \in N_-} x_j \leq l \qquad \}. \tag{5}$$

Let the inequality

$$\sum_{j \in J} x_j \leq |J| - 1 \tag{6}$$

be valid for X. There exist conic multipliers $u, v \geq 0$ such that (6) *is valid for*

$$X(u,v) = \{x \in \{0,1\}^n : x \text{ satisfies } u(4) + v(5)\}.$$

Third Case: Valid for no aggregation

Theorem 2 *There exist valid inequalities* $\sum_J x_j \leq |J| - 1$ *which are not valid for any conic combination of the constraints.*

Example:

$$X_1 = \{x \in \{0,1\}^4 : 4x_1 + 5x_2 + 6x_3 + 8x_4 \leq 14\}$$
$$X_2 = \{x \in \{0,1\}^4 : -2x_1 - 2x_2 - 3x_3 - 4x_4 \leq -6\}.$$

$$x_1 + x_2 \leq 1$$

is valid for X, neither for X_1 nor for X_2.

We prove that for all $u, v \geq 0$, there exists a solution $x \in X(u,v)$ with $x_1 = x_2 = 1$.

Outline

$$\sum_{i \in J} x_i \leq |J| - 1 \text{ valid for } X = X_1 \cap X_2.$$

1. Inequalities from X_1 or X_2 alone
   $\longrightarrow$ If $a_i, b_i \geq 0$, all inequalities fall in this category.

2. Inequalities from a combination of X_1 and X_2
   $\longrightarrow$ If $\{+1, -1\}$-coefficients in second constraint, all inequalities fall in this category.

3. Inequalities for the intersection only
   $\longrightarrow$ For general problems.

Incomplete Set Inequalities

Notation: $a(T) = \sum_{i \in T} a_i$, $b(T) = \sum_{i \in T} b_i$.

Definition 1 *I is an incomplete set if*

$$r(I) = a_0 - a(I) > 0 \quad \textit{and} \quad e(I) = b(I) - b_0 > 0,$$

called the residue and the excess.

Idea: What happens if $x_i = 1$ for all $i \in I$?

$$P_I = \{x \in \{0,1\}^{|N_- \setminus I|} : \sum_{j \in N_- \setminus I} a_j x_j \leq r(I)$$
$$\sum_{j \in N_- \setminus I} -b_j x_j \geq e(I) \quad \}.$$

Theorem 3 *Let I be an incomplete set and I^C be a covering of the solutions of P_I, then*

$$\sum_{i \in I} x_i - \sum_{j \in I^C} x_j \leq |I| - 1$$

is valid for X.

Example:

$$X_1 = \{x \in \{0,1\}^5 : 3x_1 + 2x_2 + 4x_3 + 7x_4 + 12x_5 \leq 20\}$$
$$X_2 = \{x \in \{0,1\}^5 : -7x_1 - 2x_2 - 3x_3 - 8x_4 + 9x_5 \leq 0\}$$

$I = \{5\}$, $r(I) = 8$, $e(I) = 9$.

$$P_I = \{x \in \{0,1\}^4 : \begin{array}{l} 3x_1 + 2x_2 + 4x_3 + 7x_4 \leq 8 \\ 7x_1 + 2x_2 + 3x_3 + 8x_4 \geq 9 \end{array} \}.$$

$P_I = \{(1,1,0,0),(1,0,1,0)\}$.

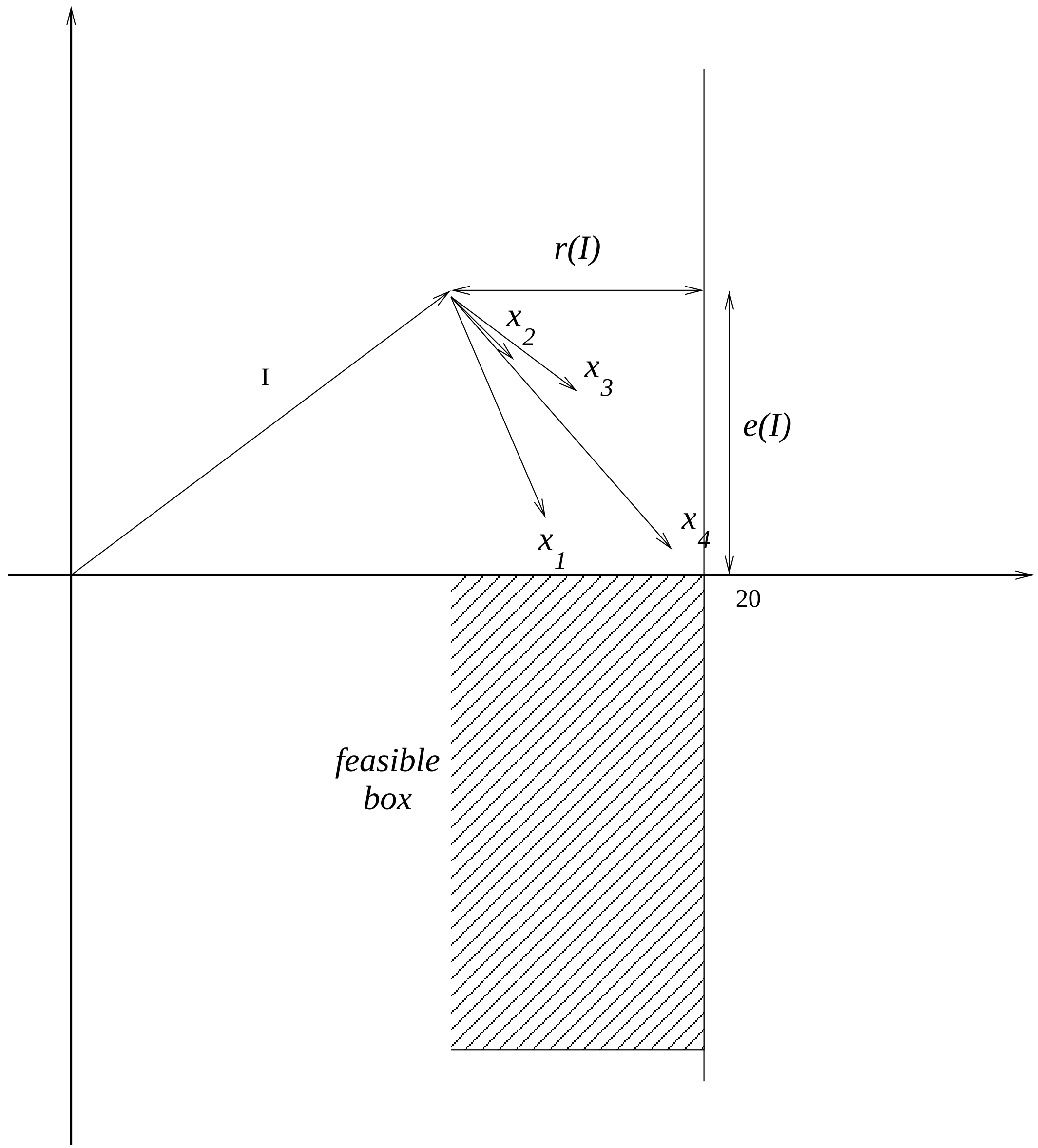
Example (continued): $P_I = \{(1,1,0,0),(1,0,1,0)\}$.
Minimal coverings of P_I: $\{1\}$, $\{2,3\}$.

$$x_5 - x_1 \leq 0$$
$$x_5 - x_2 - x_3 \leq 0$$

Why *incomplete sets*?

- Not an independence system
  Infeasible Set $\not\Rightarrow$ Every superset is infeasible

- Start with an infeasible set.
  Which conditions to complete it into a feasible set?



r(I)
I
x
2
x
3
e(I)
x
4
x
1
20
feasible
box

The strength of an incomplete set inequality

$$\sum_{i \in I} x_i - \sum_{i \in I^C} x_i \leq |I| - 1$$

Question: Suppose we have the full convex hull description of both single knapsacks, is the inequality still useful?

Relevant problem:

$$z^s = \max \sum_{i \in I} x_i - \sum_{i \in I^C} x_i$$
$$\text{s.t. } x \in \text{conv}(X_1) \cap \text{conv}(X_2).$$

Definition 2 *The strength* $s(I, I^C) = z^s - (|I| - 1)$.

Remarks:

– $0 \leq s(I, I^C) \leq 1$

– If inequality valid for X_1 or X_2, $s(I, I^C) = 0$.

Definition 3 *Let $F \subseteq N_- \setminus (I \cup I^C)$ such that $I \cup F \in X_2 \setminus X_1$.*

Theorem 4 *If*

(i) $I \cup F$ is a minimal cover for X_1

(ii) there exists $G \subset F$ and $i_0 \in I$ such that

$$b(i_0) + b(G) \geq b(I \cup F) - b_0,$$

then $s(I, I^C) \geq \frac{|G|}{|G|+1}$.

Example:

$$X_1 = \{x \in \{0,1\}^5 : 3x_1 + 2x_2 + 4x_3 + 7x_4 + 12x_5 \leq 20\}$$
$$X_2 = \{x \in \{0,1\}^5 : -7x_1 - 2x_2 - 3x_3 - 8x_4 + 9x_5 \leq 0\}.$$

$I = \{5\}$ $\quad$ $x_5 - x_1 \leq 0$ and $x_5 - x_2 - x_3 \leq 0$

	$x_5 - x_1 \leq 0$	$x_5 - x_2 - x_3 \leq 0$
F	$\{2,4\}$	$\{1,4\}$
$a(I \cup F)$	21	22
$b(I \cup F) - b_0$	-1	-6
Minimal Cover?	yes	yes
G	$\{2,4\}$	$\{1,4\}$
$b(i_0) + b(G)$	$9 - 10 = -1$	$9 - 15 = -6$
$\lvert G\rvert/(\lvert G\rvert + 1)$	$2/3$	$2/3$

$$s(x_5 - x_1 \leq 0) \geq \frac{2}{3}, \qquad s(x_5 - x_2 - x_3 \leq 0) \geq \frac{2}{3}.$$

Some additional remarks about the strength

- In practice, $s(I, I^C) \geq 1/2$ often.

- Shows the use of considering several constraints at a time.

- A more general theorem to compute the strength is available.

The mixed Case

The models:

$$X_1 = \{(x,s,t) \in \{0,1\}^n \times \mathbb{R}^2_+ : \sum_{i=1}^{n} a_i x_i \leq a_0 + s\},$$
$$X_2 = \{(x,s,t) \in \{0,1\}^n \times \mathbb{R}^2_+ : \sum_{i=1}^{n} b_i x_i \leq b_0 + t\},$$
$$X = \{(x,s,t) \in \{0,1\}^n \times \mathbb{R}^2_+ : (x,s,t) \in X_1 \cap X_2\}.$$

The method:

– Fix $s = t = 0$, obtain X_1^r, X_2^r, X^r.
– Generate an incomplete set inequality for X^r.
– Lift simultaneously the variables s and t in the inequality.

Possible to use in an arbitrary tableau

$$
\begin{array}{ccccccccccc}
x_{B1} & & +\bar{a}_{11}x_{N1} & +\cdots+ & \bar{a}_{1k}x_{Nk} & +\bar{f}_{11}s_1 & +\cdots+ & \bar{f}_{1l}s_l & = & \bar{b}_1 \\
& \ddots & \vdots & & \vdots & \vdots & & \vdots & & \\
& x_{Bl} & +\bar{a}_{l1}x_{N1} & +\cdots+ & \bar{a}_{lk}x_{Nk} & +\bar{f}_{l1}s_1 & +\cdots+ & \bar{f}_{ll}s_l & = & \bar{b}_l
\end{array}
$$

In each row i: relax s_j if $\bar{f}_{ij} > 0$ and aggregate

$$t_i = \sum_{j:\bar{f}_{ij}<0} \bar{f}_{ij}s_j.$$

$$
\begin{array}{ccccccc}
x_{B1} & & +\bar{a}_{11}x_{N1} & +\cdots+ & \bar{a}_{1k}x_{Nk} & \leq & \bar{b}_1 + t_1 \\
& \ddots & \vdots & & \vdots & & \\
& x_{Bl} & +\bar{a}_{l1}x_{N1} & +\cdots+ & \bar{a}_{lk}x_{Nk} & \leq & \bar{b}_l + t_l
\end{array}
$$

Choose two rows and generate a lifted incomplete set inequality.

Conclusion

- Useful but hard to compute inequalities.
  $\longrightarrow$ Need of good heuristics to find them.

- Very general use possible in a simplex tableau.

- Mixed case to be studied more deeply.
  $\longrightarrow$ approximate lifting, lift the variables in a different order.

- Extension to several constraints.