Intermediate IP representations using value disjunctions

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$$\max c^\top x : x \in P \cap Z^n$$

Dual methods

- based on outer description of $\mathrm{conv}(P \cap Z^n)$
- well explored
- branch-and-cut-algorithms

Primal methods

- inner descriptions of $P \cap Z^n$
- integral basis method
- reformulations with new vars

Integer Programming: State of the art

$$\max c^\top x : x \in P \cap Z^n$$

Dual methods

- based on outer description of $\operatorname{conv}(P \cap Z^n)$
- well explored
- branch-and-cut-algorithms

Primal-dual methods

- based on intermediate representations
- new variables *and* inequalities
- not explored at all

Primal methods

- inner descriptions of $P \cap Z^n$
- integral basis method
- reformulations with new vars

Consider the set $x \in \{0,1\}^8$ such that

$$8x_0 - x_1 - 2x_2 - 3x_3 - 4x_4 - 5x_5 - 6x_6 - 7x_7 \leq 0.$$

Convex hull: 13 non-trivial facets

$$\begin{array}{rrrrrrrrl}
x_0 & & & -x_3 & & -x_5 & -x_6 & -x_7 & \leq 0 \\
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5x_0 & -x_1 & -x_2 & -2x_3 & -2x_4 & -3x_5 & -4x_6 & -4x_7 & \leq 0
\end{array}$$

Intermediate representation of multi knapsack problems

An example

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\end{array}$$

An intermediate representation:

Introduce new variables for the subsets $\{1,2\}$ and $\{3,4\}$.

Reformulation

$$
\begin{aligned}
8x_0 - x_1 - 2x_2 - 3x_3 - 4x_4 - 5x_5 - 6x_6 - 7x_7 - 3x_9 - 7x_{10} &\le 0 \\
x_1 + x_2 + x_9 &\le 1 \\
x_3 + x_4 + x_{10} &\le 1
\end{aligned}
$$

Convex hull: 9 non-trivial facets

$$
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x_0 - x_2 - x_3 - x_4 - x_5 - x_7 - x_9 - x_{10} &\le 0 \\
x_0 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_9 - x_{10} &\le 0 \\
2x_0 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_9 - x_{10} &\le 0 \\
2x_0 - x_2 - x_3 - x_4 - x_5 - x_6 - 2x_7 - x_9 - 2x_{10} &\le 0 \\
+ x_3 + x_4 + x_{10} &\le 1 \\
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How to obtain intermediate representations?

starting point: a knapsack relaxation, for instance

$$P = \text{conv}\{x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \le b\}$$

one tool: value disjunctions

Partition $N = \{1, \dots, n\}$ into subsets $N_1, \dots, N_K$.

Reformulation based on N_i

Let $\{d_1, \dots, d_{n_i}\} = \{\sum_{i \in S} a_i \mid S \subseteq N_i\}$. For each value d_k we introduce a binary variable $y^{N_i,k}$.

linking constraints:

$$\sum_{j \in N_i} a_j x_j = \sum_{k=1}^{n_i} d_k y^{N_i,k}$$

packing constraints:

$$\sum_{k=1}^{n_i} y^{N_i,k} \le 1$$

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An example

$$3x_1 + 3x_2 + 3x_3 + 4x_4 + 5x_5 \leq 9.$$

Two blocks and six new variables

Block N_1	$\{1, 2, 3\}$	Values: 3,6,9	New variables: y_1, y_2, y_3
Block N_2	$\{4, 5\}$	Values: 4,5,9	New variables: z_1, z_2, z_3

Reformulation

$$3y_1 + 6y_2 + 9y_3 + 4z_1 + 5z_2 + 9z_3 \leq 9$$
$$3x_1 + 3x_2 + 3x_3 = 3y_1 + 6y_2 + 9y_3$$
$$4x_4 + 5x_5 = 4z_1 + 5z_2 + 9z_3$$
$$y_1 + y_2 + y_3 \leq 1$$
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$$y_1 + y_2 + y_3 \leq 1$$
$$z_1 + z_2 + z_3 \leq 1$$

How to obtain intermediate representations?

Reformulations based on value disjunctions

- generalizes single variable multiplication;

$$x_1 + x_2 = z_1 + 2z_2 \iff z_2 = x_1 x_2$$

- is more compact than full enumeration;

$$\sum_{i\in S} x_i \to \sum_{T\subseteq S} z_T = 1 \quad \text{versus} \quad \sum_{i\in S} x_i = \sum_{k=1}^{|S|} k z_k;\ \sum_k z_k \leq 1$$

- avoids projection;
- is much stronger than reformulation $\sum_{i\in S} x_i - z = 0,\ z \in \mathbf{Z}_+$;
- yields a much nicer description than using the binary expansion of z, i.e.,

$$\sum_{i\in S} x_i = \sum_{k=0}^{\lfloor \log(|S|) \rfloor} 2^k z_k, \quad z_k \in \{0,1\}.$$

An example

$$3x_1 + 3x_2 + 3x_3 + 3x_4 + 4x_5 + 7x_6 + 8x_7 + 9x_8 + 13x_9 + 15x_{10} \leq 45.$$

Formulation	Equations	# Facets
original		328
integer expansion	$x_1 + x_2 + x_3 + x_4 = z$	328
binary expansion	$x_1 + x_2 + x_3 + x_4 = z_1 + 2z_2 + 4z_3$	217
value disjunction	$x_1 + x_2 + x_3 + x_4 = z_1 + 2z_2 + 3z_3 + 4z_4$ $z_1 + z_2 + z_3 + z_4 \leq 1$	77

Structural theorem for value disjunctions

value disjunction polytope

$$V_i = \text{conv}\left\{ (x^{N_i}, y^{N_i}) \in \{0,1\}^{|N_i|} \times \{0,1\}^{n_i} : \sum_{j \in N_i} a_j x_j = \sum_{k=1}^{n_i} a(y^{N_i,k}) y^{N_i,k} \quad \sum_{k=1}^{n_i} y^{N_i,k} \leq 1 \right\}.$$

aggregated polytope

$$Q = \text{conv}\left\{ y \in \{0,1\}^{n_1+\cdots+n_K} : \sum_{i=1}^{K} \sum_{k=1}^{n_i} a(y^{N_i,k}) y^{N_i,k} \leq b \quad \sum_{k=1}^{n_i} y^{N_i,k} \leq 1 \quad \forall i \right\}$$

Theorem (structural theorem)

$$P = \{\, x \in [0,1]^n : \textit{there is } y \in [0,1]^{n_1+\cdots+n_K} \textit{ such that } (x^{N_i}, y^{N_i}) \in V_i \textit{ for } i = 1, \ldots, K \textit{ and } y \in Q \,\}.$$

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The value disjunction polytope V_i: The cardinality case

Theorem

$$V_i = \text{conv}\{(x^{N_i}, y^{N_i}) \in \{0,1\}^{|N_i|} \times \{0,1\}^{n_i} : \sum_{j \in N_i} x_j = \sum_{k=1}^{n_i} k y^{N_i,k}, \quad \sum_{k=1}^{n_i} y^{N_i,k} \leq 1\}.$$

is completely described by non-negativity constraints and:

$$\sum_{j \in N_i} x_j = \sum_{k=1}^{n_i} k y^{N_i,k}$$

$$\sum_{j \in T} x_j - \sum_{k=1}^{|T|} k y_k - \sum_{k=|T|+1}^{n_i} |T| y_k \leq 0 \qquad \text{for } \emptyset \neq T \subset N_i$$

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Theorem

The separation problem over V_i can be solved in polynomial time.

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The knapsack with three distinct coefficients

The problem

$$\sum_{j\in N_1} \mu x_j + \sum_{j\in N_2} \lambda x_j + \sum_{j\in N_3} \sigma x_j \le \beta,$$

An extended formulation

$$\sum_{j\in N_1} \mu x_j + \sum_{j\in N_2} \lambda x_j + \sum_{j\in N_3} \sigma x_j \le \beta$$

$$\sum_{j\in N_i} x_j = \sum_{k=1}^{|N_i|} k y_k^i \qquad \text{for } i = 1,2,3$$

$$\sum_{k=1}^{|N_i|} y_k^i \le 1 \qquad \text{for } i = 1,2,3$$

$$x \in \{0,1\}^{|N_1|+|N_2|+|N_3|}$$

$$y^i \in \{0,1\}^{|N_i|} \qquad \text{for } i = 1,2,3.$$

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The aggregated polyhedron

$$\mu \sum_{k=1}^{|N_1|} k y^{N_1,k} + \lambda \sum_{k=1}^{|N_2|} k y^{N_2,k} + \sigma \sum_{k=1}^{|N_3|} k y^{N_3,k} \leq \beta$$

$$\sum_{k=1}^{|N_i|} y^{N_i,k} \leq 1 \qquad \text{for } i = 1,2,3$$

$$y^{N_i} \in \{0,1\}^{|N_i|} \qquad \text{for } i = 1,2,3.$$

- Let $\{v^1, \ldots, v^p\} \subseteq \{0,1\}^{|N_1|+|N_2|+|N_3|}$ be all the vertices of the aggregated polyhedron.
- Notice that $p \leq (1+|N_1|)\cdot(1+|N_2|)\cdot(1+|N_3|)$.

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- Let $\{v^1, \ldots, v^p\} \subseteq \{0,1\}^{|N_1|+|N_2|+|N_3|}$ be all the vertices of the aggregated polyhedron.
- Notice that $p \leq (1+|N_1|)\cdot(1+|N_2|)\cdot(1+|N_3|)$.

The knapsack with three distinct coefficients

The aggregated polyhedron

$$\mu \sum_{k=1}^{|N_1|} k y^{N_1,k} + \lambda \sum_{k=1}^{|N_2|} k y^{N_2,k} + \sigma \sum_{k=1}^{|N_3|} k y^{N_3,k} \leq \beta$$

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Theorem

The complete facet description in an extended space is:

$$
\begin{aligned}
y &= \sum_{j=1}^{p} v^j z_j \\
\sum_{j=1}^{p} z_j &= 1 \\
z_j &\geq 0 && \text{for } j = 1, \ldots, p \\
\sum_{j \in N_i} x_j^{N_i} &= \sum_{k=1}^{n_i} k y^{N_i,k} && \text{for } i = 1, 2, 3 \\
\sum_{j \in T} x_j^{N_i} &\geq \sum_{\substack{k \in \{1,\ldots,n_i\}: \\ |T|+k > n_i}} (|T| + k - n_i) y^{N_i,k} && \text{for } i = 1, 2, 3 \text{ and } \emptyset \neq T \subset N_i \\
x &\in \mathbf{R}^{|N_1|+|N_2|+|N_3|} \\
y &\in \mathbf{R}^{|N_1|+|N_2|+|N_3|} \\
z &\in \mathbf{R}^{p}.
\end{aligned}
$$

The simplification effect of branching

Initial Problem

2 constraints and 12 variables

13083 facets

1. Fix $x_2 = 0,\ x_6 = 0$
   690 facets
2. Fix $x_2 = 0,\ x_6 = 1$
   425 facets
3. Fix $x_2 = 1,\ x_6 = 0$
   91 facets
4. Fix $x_2 = 1,\ x_6 = 1$
   541 facets
5. Total : 1747 facets

Comparing Variable Branching with Value Disjunction

$\binom{12}{2}$ possible choices of x_i, x_j $\binom{12}{3}$ possible choices of x_r, x_s, x_t

Compute the number of facets for all four cases

$x_i = 0, x_j = 0,\ x_i = 1, x_j = 1$ $x_r + x_s + x_t = 0,\ x_r + x_s + x_t = 1$

$x_i = 1, x_j = 0,\ x_i = 0, x_j = 1$ $x_r + x_s + x_t = 2,\ x_r + x_s + x_t = 3$

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Experiments with branching

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Claim

It is efficient to use value disjunction on a set of variables that are similar (that have the same structure).

Ranking formula

We create a ranking formula that allows us to say whether a triple of variables is structured or not.

$$\begin{pmatrix} 7 & 8 & 7 \\ 11 & 9 & 10 \end{pmatrix} \text{ has a good ranking}$$

$$\begin{pmatrix} -23 & 12 & -6 \\ 4 & -1 & -14 \end{pmatrix} \text{ has a bad ranking}$$

Branching on value disjunctions vs. 2-variable branching

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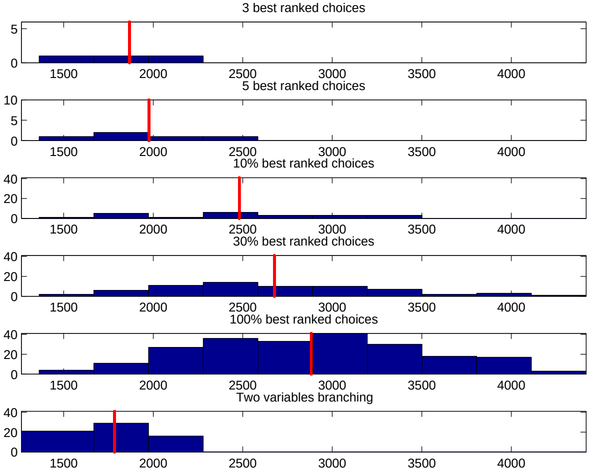
Branching on value disjunctions vs. 2-variable branching ("unstructured")

Histograms of the total number of facets in the subproblems

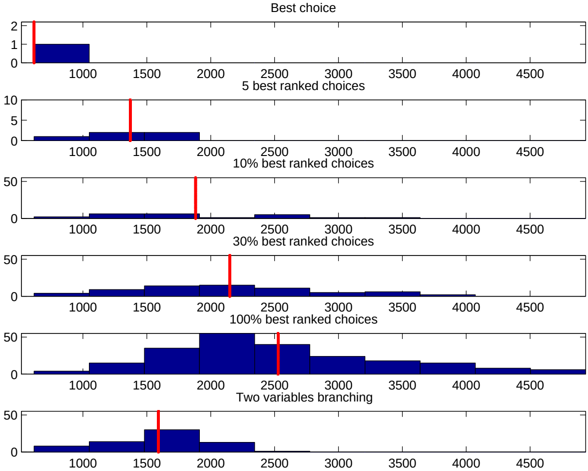
$$\begin{array}{rrrrrrrrrrrrrl} 11 & -7 & 9 & 10 & -2 & 7 & 14 & -15 & 4 & -5 & -2 & -19 & \leq 0 \\ 6 & 18 & -4 & -9 & 17 & -11 & 5 & -12 & 5 & 3 & -18 & 7 & \leq 0 \end{array}$$



Branching on value disjunctions vs. 2-variable branching ("structured")

Histograms of the total number of facets in the subproblems

$$\begin{array}{rrrrrrrrrrrrl} 7 & 6 & 7 & 15 & -21 & -15 & -23 & -12 & 12 & -6 & 11 & 10 & \leq 0 \\ 10 & 10 & 9 & -21 & 4 & -3 & 4 & 13 & -1 & -14 & 2 & -6 & \leq 0 \end{array}$$



Branching for market split and mas instances

			CPLEX 9.1		Value Disjunctions	
Name	Rows	Cols	Nodes (10^6)	Time (s)	Nodes (10^6)	Time (s)
corn535-1	5	35	13.8	2 431	3.8	809
corn535-2	5	35	11.9	2 084	4.2	865
corn535-3	5	35	17	2 946	9.8	1 970
corn540-4	5	40	321	55 918	105	20 873
corn540-5	5	40	231	39 787	87	17 267
corn540-6	5	40	188	30 532	97	19 162
corn650-7	6	50	***	***	20400	4.4 M
mas74	13	151	5.3		0.7	
mas76	12	151	0.413		0.093	

Computation times in CPU seconds on a Sun Fire V890 with 1200 MHz UltraSPARC-IV processors