Separation for the two-row problem

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Joint work with Laurent Poirrier (Liège)

Outline

- Cuts from two rows of the simplex tableau : our notation
- Generating sparse cuts
- An algorithm for the separation problem
- Preliminary computational results
- Conclusion and future work

Relaxation of MIP

Simplex Tableau

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$r_{m,1}x_1$	$\cdots +$	$r_{m,k}x_k$	$+$	$r_{m,k+1}s_{k+1}$	$\cdots +$	$r_{m,n}s_n$
$s_{B_{m+1}}$	$=$	f_{m+1}	$+$	$r_{m+1,1}x_1$	$\cdots +$	$r_{m+1,k}x_k$	$+$	$r_{m+1,k+1}s_{k+1}$	$\cdots +$	$r_{m+1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
s_{B_p}	$=$	f_p	$+$	$r_{p,1}x_1$	$\cdots +$	$r_{p,k}x_k$	$+$	$r_{p,k+1}s_{k+1}$	$\cdots +$	$r_{pn}s_n$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+$
2. $s_{B_{m+1}}, \ldots, s_{B_p} \in \mathbb{R}_+$
3. $x_1, \ldots, x_k \in \mathbb{Z}_+$
4. $s_{k+1}, \ldots, s_n \in \mathbb{R}_+$

Solution is 'fractional', i.e. $f_1, \ldots, f_m$ are not all integer.

Relaxation of MIP

Relaxation Step 1 : Drop Some Constraints

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
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$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
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Relaxation of Simplex Tableau

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x_{B_2}	$=$	f_2	$+$	$r_{2,1}x_1$	$\cdots +$	$r_{2,k}x_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

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$(f_1, f_2) \notin \mathbb{Z}^2$.

Relaxation of MIP

Relaxation Step 2 : Drop Integrality Requirement

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
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Relaxation of MIP

Relaxation Step 2 : Drop Integrality Requirement

Basic Variable		rhs		Columns Corresponding to Continous Variables						
x_{B_1}	$=$	f_1	$+$	$r_{1,1}s_1$	$\cdots +$	$r_{1,k}s_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
x_{B_2}	$=$	f_2	$+$	$r_{2,1}s_1$	$\cdots +$	$r_{2,k}s_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

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Continuous Group Relaxation

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x_{B_2}	$=$	f_2	$+$	$r_{2,1}s_1$	$\cdots +$	$r_{2,k}s_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

1. $x_{B_1}, x_{B_2} \in \mathbb{Z}_+ \xrightarrow{\text{Relaxation}} x_{B_1}, x_{B_2} \in \mathbb{Z}$
2. $s_1, \dots, s_k,\ s_{k+1}, \dots, s_n \in \mathbb{R}_+$

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The valid inequalities for the above are valid for the original MIp

Model studied in Andersen, Louveaux, Weismantel, Wolsey, IPCO2007 (for the finite case), Cornuéjols and Margot, 2009.

Related to Group Relaxation of Gomory and Johnson (1972), Johnson (1974).

The 2 row-model

The model $x = f + Rs$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j=1}^{k} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j, \qquad x_1, x_2 \in \mathbb{Z}, s_j \in \mathbb{R}_+$$

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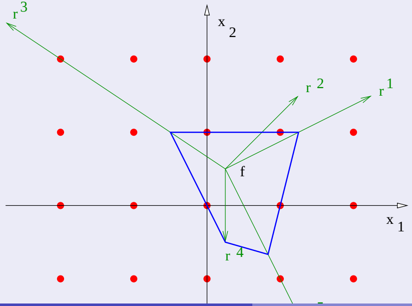
The geometry

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5$$



$$2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7} s_5 \geq 1$$

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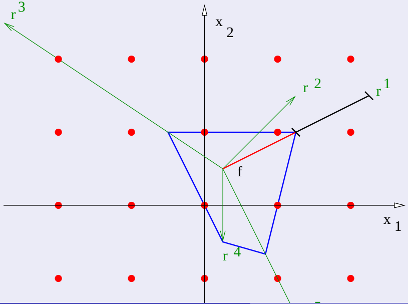

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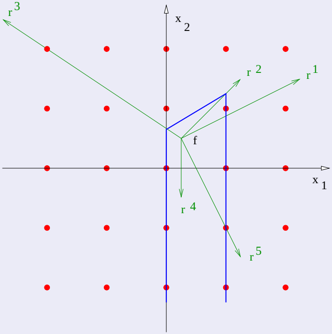
Sparsity of the cuts

The initial model

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j=1}^{k} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j,$$

$x_1, x_2 \in \mathbb{Z}, s_j \in \mathbb{R}_+$

Most variables get a nonzero coeff in the cut !
At most one direction gets a 0 coefficient
($\Rightarrow$ Split).



The cuts generated from the plain model are not sparse.

Selecting the two rows (I)

A way to select the two rows is to create cuts as sparse as possible.

$$\begin{array}{ccccc} x_1 & & +\bar{a}_{11}s_1 + \cdots + \bar{a}_{1k}s_k & = & \bar{b}_1 \\ & \ddots & \vdots \ddots & & \vdots \\ & & x_m + \bar{a}_{m1}s_1 + \cdots + \bar{a}_{mk}s_k & = & \bar{b}_m \end{array}$$

- Out of k nonbasic variables, one can choose *a priori* $m-p$ columns (of rank $m-p$) to be set to 0
- We must consider the lattice

$$\mathcal{L} = \{u \in \mathbb{Z}^m \mid \begin{array}{l} \bar{a}_{11}u_1 + \cdots + \bar{a}_{m1}u_m = 0 \\ \vdots \\ \bar{a}_{1,m-p}u_1 + \cdots + \bar{a}_{m,m-p}u_m = 0 \end{array} \}$$

  and obtain p rows that have $m-p$ additional zeros.
- Minor detail : we must do the computation in rationals

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Selecting the two rows (I)

- A way to compute a solution to the system is to find a short vector in the lattice.
- We use the method of Aardal, Hurkens, Lenstra by computing an LLL reduced basis of the lattice

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ M\bar{a}_{11} & \cdots & M\bar{a}_{m1} \\ \vdots & & \vdots \\ M\bar{a}_{1,m-p} & \cdots & M\bar{a}_{m,m-p} \end{pmatrix}$$

  Short vectors of this lattice have 0's in the last $m-p$ entries and therefore provide an element of the lattice $\mathcal{L}$.
- We can include more columns (if not all) and let LLL find the k variables set to 0.

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ M\bar{a}_{11} & \cdots & M\bar{a}_{m1} \\ \vdots & & \vdots \\ M\bar{a}_{1,k} & \cdots & M\bar{a}_{m,k} \end{pmatrix}$$

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The separation problem for the 2-row model

In the following, we fix the model from which we want to generate a cutting plane.

$$P_I = \left\{ x_1, x_2 \in \mathbb{Z}, s \in \mathbb{R}^k_+ \,\middle|\, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j=1}^{k} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j \right\}.$$

Given a point $(\hat{x}, \hat{s}) \in \mathbb{R}^2 \times \mathbb{R}^k$, we want to

- either state that $(\hat{x}, \hat{s}) \in \text{conv}(P_I)$
- or find the valid inequality for $\text{conv}(P_I)$ that is most violated by $(\hat{x}, \hat{s})$.

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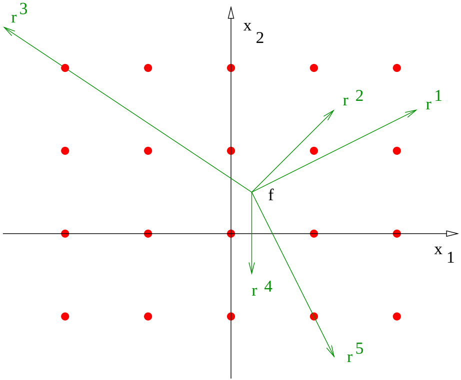
The polar system for the 2-row model

The polar of a polyhedron

Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $Q \subseteq \mathbb{R}^n$ its polar.
There is a correspondence between

Extreme point $x \in P$	and	Facet of Q of the type $x^T a \geq 1$
Extreme ray $x \in P$	and	Facet of Q of the type $x^T a \geq 0$
Facet of P of the type $a^T x \geq 1$	and	Extreme point $a \in Q$
Facet of P of the type $a^T x \geq 0$	and	Extreme ray $a \in Q$

The polar system for the 2-row model

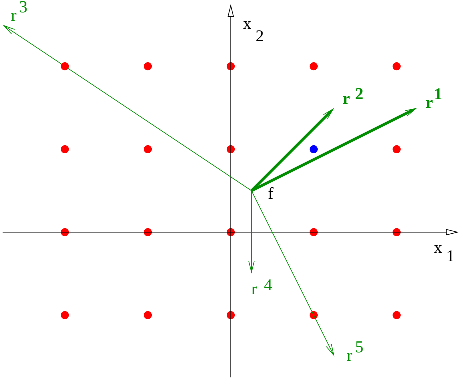


What are extreme points of conv(P_I)?

$$x = f + RS$$

They correspond to points $(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^k_+$
such that support$(s) \leq 2$.

The polar system for the 2-row model



$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} + \frac{1}{4} r^1 + \frac{1}{4} r^2$$

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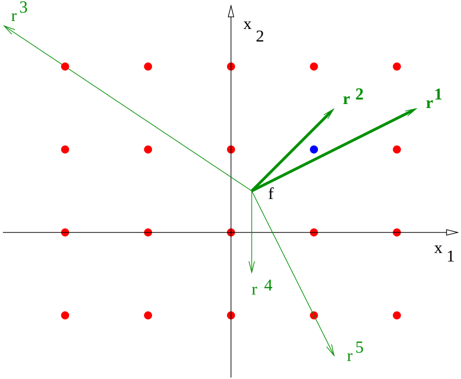
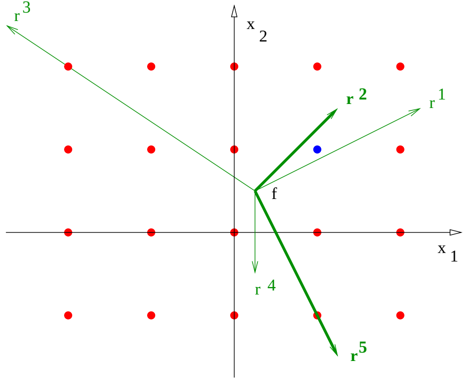


$$\begin{pmatrix} \color{blue}{1} \\ \color{blue}{1} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} + \frac{1}{4}\color{green}{r^1} + \frac{1}{4}\color{green}{r^2}$$

The polar

$$\frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 \geq 1$$

The polar system for the 2-row model



$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} + \frac{2}{3} r^2 + \frac{1}{12} r^5$$

The polar

$$\frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 \geq 1$$
$$\frac{2}{3}\alpha_2 + \frac{1}{12}\alpha_5 \geq 1$$

Complexity of writing the polar

- For each cone, compute the integer hull.
- For each integer point in each integer hull, compute the representation in the given cone and generate one inequality for the polar
- Quadratic complexity in the number of rays for the number of cones
- Polynomial number of integer vertices in each cone (but may be large if the numbers involved are large)
- The rays must be available in rationals

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Reducing the complexity of the number of cones to consider

Ordering the cones

Let $\mathcal{C}$ be the set of cones $f + \text{cone}\{r^i, r^j\}$. In 2D, we can order the cones by

- considering only the cones that have no proper subcones.
- ordering the rays anti-clockwise (for example) $r^1, \ldots, r^k$.

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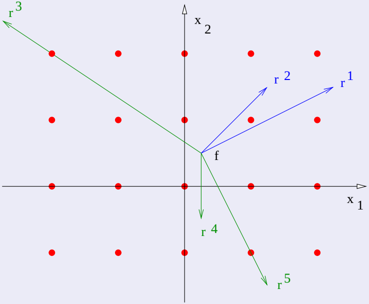

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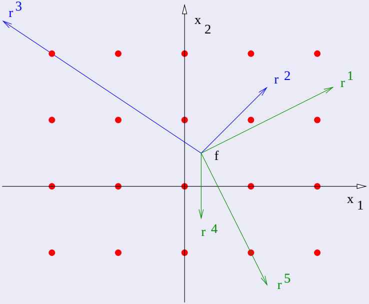

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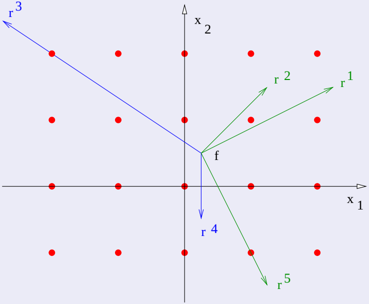

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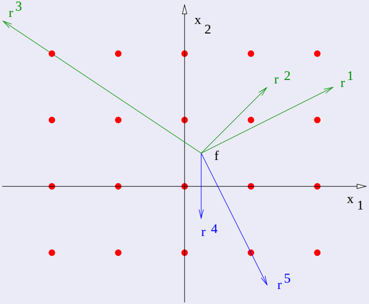

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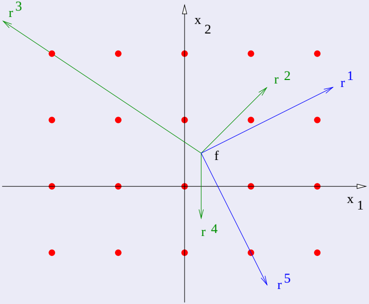

Reducing the complexity of the number of cones to consider

Theorem

For each i, j, let $\mathcal{X}_{i,j}$ be the set of vertices of $\text{conv}((f + \text{cone}(r^i, r^j) \cap \mathbb{Z}^2)$.
Consider the polar

$$Q = \{\alpha \in \mathbb{R}^k_+ \mid \forall i, j, \forall x \in \mathcal{X}_{i,j} \text{ s.t. } x = f + s_i r^i + s_j r^j, s_i, s_j \geq 0 \\ s_i \alpha_i + s_j \alpha_j \geq 1 \quad \}$$

Consider the set

$$\bar{Q} = \{\alpha \in \mathbb{R}^k_+ \mid \forall i, \forall x \in \mathcal{X}_{i,i+1} \text{ s.t. } x = f + s_i r^i + s_{i+1} r^{i+1}, s_i, s_{i+1} \geq 0 \\ s_i \alpha_i + s_{i+1} \alpha_{i+1} \geq 1 \\ \forall i \text{ s.t. } r^i = \lambda r^{i-1} + \mu r^{i+1}, \lambda, \mu \geq 0 \\ \alpha_i \leq \lambda \alpha_{i-1} + \mu \alpha_{i+1} \quad \}$$

An optimal solution to $\begin{array}{ll} \min & c^T \alpha \\ \text{s.t.} & \alpha \in \bar{Q} \end{array}$ is an optimal solution to $\begin{array}{ll} \min & c^T \alpha \\ \text{s.t.} & \alpha \in Q \end{array}$

Reducing the number of integer points to generate

A facet of the 2-row problem is tight at at most four integer points.
Is it necessary to generate all vertices of all cones ?

Sketch of the algorithm

- Generate a few trivial integer points
  - All four rounded values of f
  - One potential integer point on each ray $f + \lambda r^i$
  - For each integer point, determine the cone in which it lies and write the corresponding constraint
- Determine an optimal solution α of this incomplete polar
- Check geometrically whether α is valid
- If yes, done !
- If not, determine an integer point that violates α
  Determine the corresponding cone, generate the corresponding inequality in the polar and start again.

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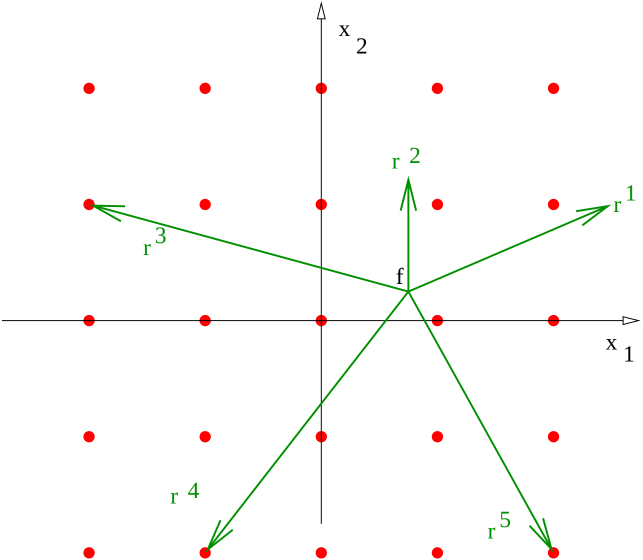

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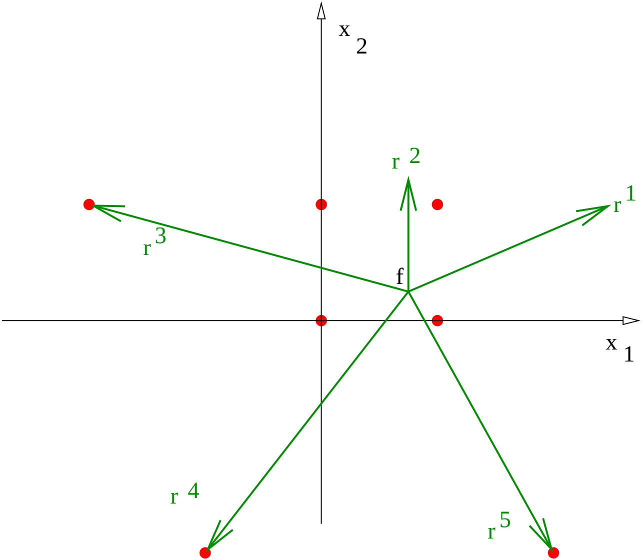

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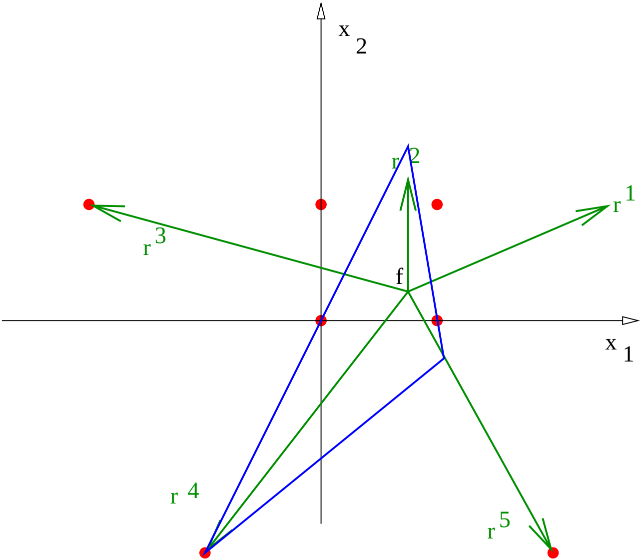

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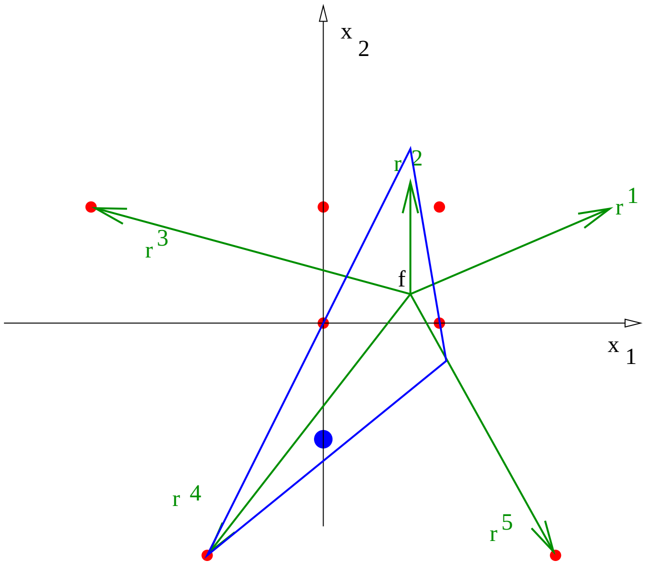

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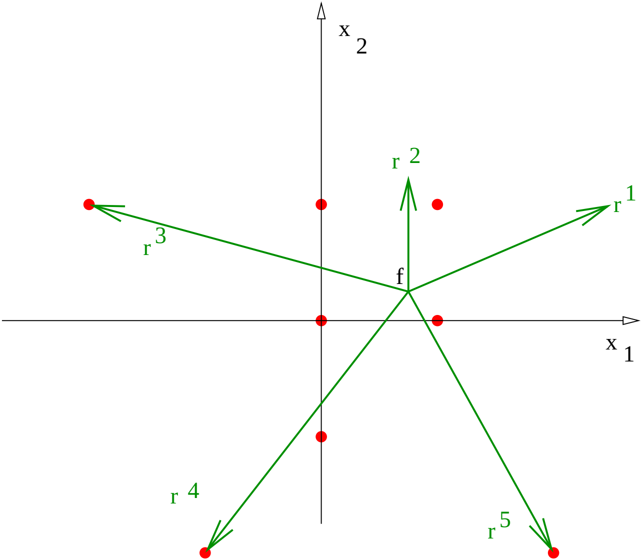

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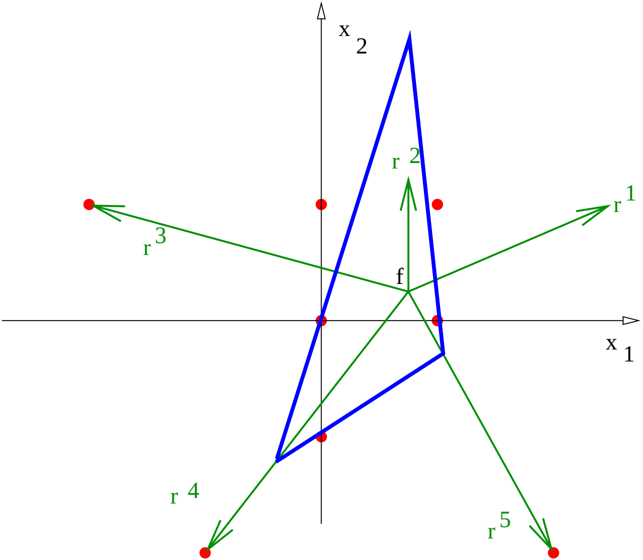

How to check the validity of an inequality ?

- Tight integer points : It could be that the inequality is tight at more than four integer points
  $\Rightarrow$ there must be an integer point in the interior
- If the inequality is tight at three integer points
  there could be an integer point in the interior of the convex hull of the three integer points.
  $\rightarrow$ easily checked through the determinant of the underlying triangle

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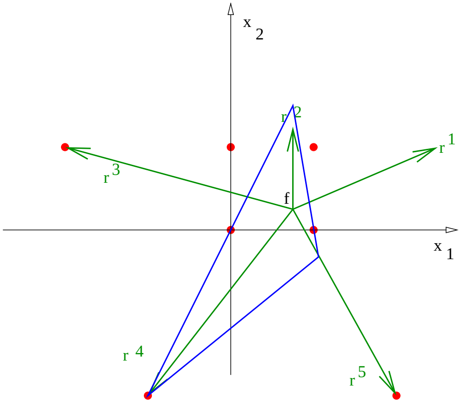

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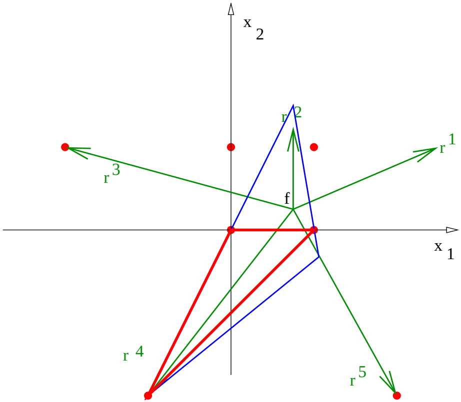

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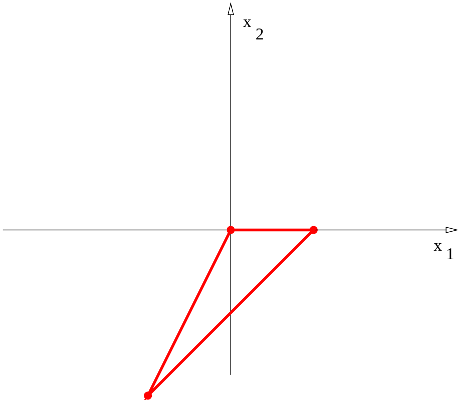

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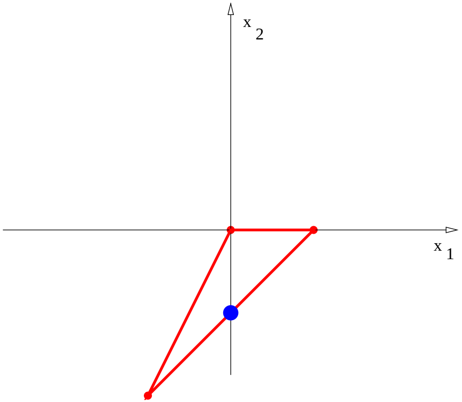

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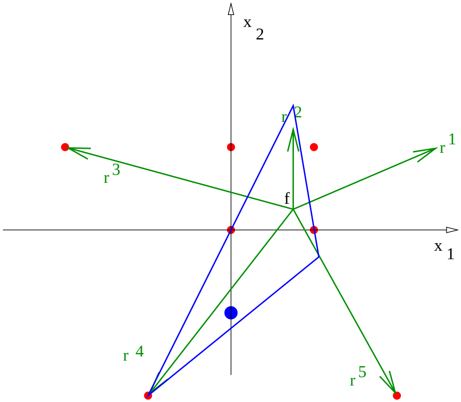

How to check the validity of an inequality ?

Assume that the underlying triangle or quadrilateral is unimodular.
There could still be some integer point in the interior of the inequality.

Lemma

If the underlying triangle is unimodular, there are only three points to be checked to ensure validity.

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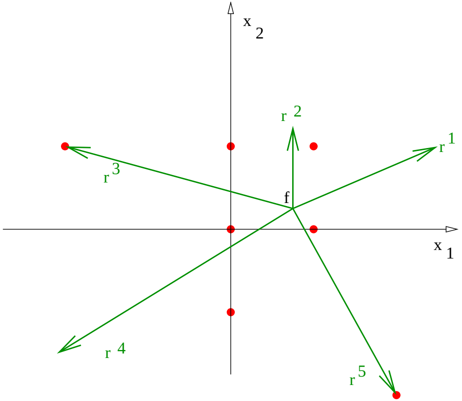

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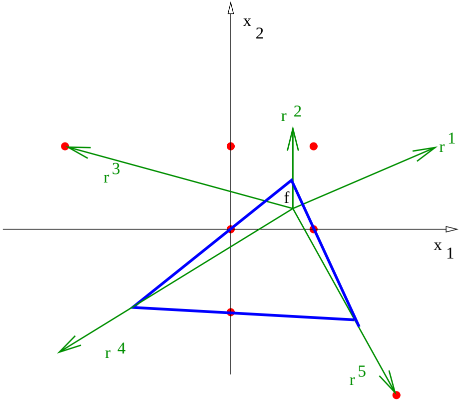

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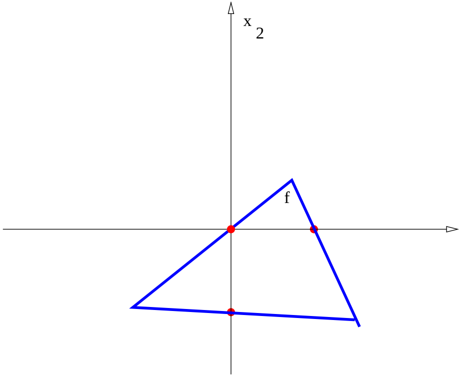

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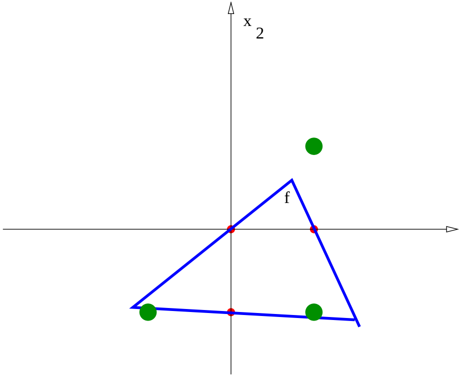

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If the underlying triangle is unimodular, there are only three points to be checked to ensure validity.

Summary of the algorithm

(1) In the polar, generate an inequality for each initial integer point considered

(2) Generate an inequality $\alpha_i \leq \lambda\alpha_{i-1} + \mu\alpha_{i+1}$ for each ray

(3) Solve the incomplete polar

(4) Search for the tight integer points $x_1, \ldots, x_n$

(5) Check whether $\text{conv}\{x_1, \ldots, x_n\}$ contains other integer points

(6) Check whether α is valid

Preliminary computational results

Name	Gap closed (%)	Time (s)	N cuts	N iter
10teams	28.57	153.11	18	18
bell3a	64.15	1.62	24	34
bell5	73.16	1.08	39	79
blend2	18.34	0.73	2	4
dcmulti	55.67	31.51	33	34
egout	24.35	0.04	25	25
fiber	0.59	0.04	9	11
fixnet6	7.71	9.86	11	13
gen	14.83	0.61	33	38
gesa2	30.44	0.32	47	47
gesa3	24.9	0.03	14	15
gt2	14.05	0.17	44	52
harp2	3.66	0.75	10	13
khb05250	88.53	0.89	46	51
lseu	39.15	1.12	42	49
markshare1	0	107.61	130	145
misc07	0.6	701.67	20	33

Selection of the rows (II) : heurisitic that considers sparsity and avoiding numerically instable cuts.

Remark : 98% of the time is spent in solving the rational LP (the polar)

Using LLL to generate pairs of sparse rows

Name	Initial heur.		Using LLL	
	N cuts	Gap cl.	N cuts	Gap cl.
bell3a	24	64.15	12	70.74
bell5	39	73.16	32	48.49
egout	25	24.35	72	96.15
lseu	42	39.15	20	45.47

In general, the cuts are numerically more stable.

Conclusions and future work

Conclusions

- It is extremely fast to separate once the model is fixed
- The gap closed by the 2-row model is not negligible but most of it is achieved by split cuts (i.e. 1-row cuts). There is a need to strenghten the cuts (with lifting for example).
- There should be more in the n-row models but the row selection is hard.

Future work

- Extension to m rows, $m \geq 3$
  - Consider the cones that have no proper subcones
  - No ordering of the cones, complexity is not reduced
  - Checking validity or finding a violated point is trickier
- Go beyond the Kelley scheme
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