Synchronization and Balancing on the N-Torus ⋆

L. Scardovi *, A. Sarlette, R. Sepulchre

Department of Electrical Engineering and Computer Science, University of Liège, Montefiore Institute, B-4000 Liège, Belgium

Abstract

In this paper, we study the behavior of a network of N agents, each evolving on the circle. We propose a novel algorithm that achieves synchronization or balancing in phase models under weak connectedness assumptions on the (possibly time-varying and unidirectional) communication graphs. The global convergence analysis on the N-torus is a distinctive feature of the present work with respect to previous results that have focused on convergence in the Euclidean space.

Key words: Synchronization, Distributed algorithms, Phase models, Consensus algorithms, Stability

1 Introduction

Over the past decade, particular attention has been devoted to the study of collective problems where interacting agents must reach a common objective under information and communication constraints. These problems arise in a

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* Corresponding author: Phone:+32 4 366 29 87, Fax:+32 4 366 29 89.
Email addresses: `l.scardovi@ulg.ac.be` (L. Scardovi), `sarlette@montefiore.ulg.ac.be` (A. Sarlette), `r.sepulchre@ulg.ac.be` (R. Sepulchre).

variety of disciplines including physics, biology, computer science and systems and control theory. Analysis and design efforts have been devoted to understand how a group of moving agents (e.g. flocks of birds, schools of fish or autonomous robots) can reach a consensus without an external reference and in a decentralized way. Applications include formation control of autonomous vehicles [2,14] and sensor networks [5,6]. In physics, synchronization phenomena in populations of coupled oscillators have received a lot of attention [4,15,16]. These phenomena have been studied mainly by means of phase models giving rise to the celebrated *Kuramoto model* and in these last years the related dynamics have been investigated by means of system theoretic tools [12,3,8].

In those applications, the collective design can be formalized as the design of a decentralized algorithm for the collective optimization of a suitable cost function characterizing a common objective [11]. The natural –e.g. gradient-based – optimization algorithms require all-to-all information exchange because the cost function depends on the entire state. In the present paper we call such algorithms *global information algorithms*. However the communication constraints restrict the information available to a given agent at a given instant of time. In the present paper we call the algorithms that fulfill the communication constraints *local information algorithms*. The optimization based design of local information algorithms either requires to constrain the cost function in accordance with the communication constraints or to approximate the global information algorithm with a local one. The first solution –adapting the cost function – is systematic but challenging when the communication constraints are uncertain and might change over time, which is the typical situation encountered in practice. The present paper focuses on the second solution, which consists in approximating the global information algorithm.

We focus on the distributed stabilization of a phase model in continuous and discrete time. Because each phase variable evolves on the circle S^1, the total state-space is the $N-$torus $T^N = S^1 \times \ldots \times S^1$. The global convergence analysis on the N-torus is a distinctive feature of the present work with respect to previous synchronization results [9],[8],[3] that have focused on convergence in the Euclidean space, considering the present problem either by local linearization or by restriction of the initial conditions to a subdomain diffeomorphic to the Euclidean space.

The paper is organized as follows. In the next section, we formalize the problem of *synchronization and balancing on the N-Torus*. In Section 3 we review the problem of reaching a consensus in the Euclidean space in a distributed setting and in Section 4 a natural extension to the N-Torus is provided. Section 5 and Section 6 present local information algorithms and global convergence analysis of the proposed decentralized algorithms is established. Finally, in Section 7, we conclude with some observations and perspectives for future research.

2 Synchronization and balancing on the N-Torus

Consider N autonomous agents evolving on the circle, each agent is represented by its state $\theta_k \in S^1, \;\; k = 1, \ldots, N$. The total state space is the N-torus $T^N = S^1 \times \ldots \times S^1$, we will indicate by $\theta \in \mathbb{T}^N$ the state of the overall system. We consider algorithms that only use relative information such as phase differences. The resulting state space is then the *quotient shape space* T^N/S^1 where all states differing by a rigid rotation are identified. A *synchronized* state is a configuration in which all the agents lie at the same position on the circle. In contrast, a *balanced* state is reached when the agents are "dispersed" on the circle. The concept of synchronization and balancing is formalized by the definition of the centroid

$$p_\theta = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k} = |p_\theta| e^{i\psi} \in \mathbb{C}. \tag{1}$$

The parameter $|p_\theta|$ is a measure of synchrony of the phase variables θ. It is maximal when all phases are synchronized (identical). It is minimal when the phases balance to result in a vanishing centroid. Hence synchronization and balancing correspond to maximizing or minimizing the cost function

$$V(\theta) = \frac{N}{2}|p_\theta|^2. \tag{2}$$

Its gradient is computed as

$$\frac{\partial V}{\partial \theta_k} =< p_\theta, ie^{i\theta_k} >= \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_k)\,, \tag{3}$$

where the inner product $< \cdot, \cdot >$ is defined by $< z_1, z_2 >= \mathrm{Re}\{\bar{z}_1 z_2\}$ for $z_1, z_2 \in \mathbb{C} \approx \mathbb{R}^2$ (which is an inner product over the real numbers). A continuous-time gradient algorithm associated to the cost function (2) is

$$\dot{\theta}_k = -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_k) = -K < p_\theta, ie^{i\theta_k} >, \tag{4}$$

for $k = 1, \ldots, N$, where the sign of the parameter K determines a descent or ascent algorithm for the cost (2). We report here a result in [12] that provides a characterization of the critical points of (4):

Theorem 1 *The potential $V(\theta) = \frac{N}{2}|p_\theta|^2$ reaches its unique minimum when $p_\theta = 0$ (balancing) and its unique maximum when all phases are identical (synchronization). All other critical points of $V(\theta)$ are isolated in the shape manifold T^N/S^1 and are saddle points. The phase model (4) forces convergence of all solutions to the critical set of $V(\theta)$. If $K < 0$, then only the set of synchronized states is asymptotically stable and every other equilibrium is unstable. If $K > 0$, then only the set of balanced states is asymptotically stable and every other equilibrium is unstable.* □

Because the dynamics (4) evolve in the shape manifold T^N/S^1, it is worth noting that the conclusions of Theorem 1 are equivalently stated in a rotating frame, that is, for the phase model

$$\dot{\theta}_k = \omega - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_k) = \omega - K < p_\theta, ie^{i\theta_k} >, \quad \omega \in \mathbb{R}. \tag{5}$$

This all-to-all model is the most frequently studied coupling in the literature of coupled oscillators [4,16,15]. It is a particular case of the celebrated *Kuramoto model* where each oscillator is modeled by a phase variable $\theta_k \in S^1$ that, in the absence of coupling, obeys the trivial dynamics $\dot{\theta}_k = \omega_k$ where ω_k is the natural frequency of oscillator k. Its application in the context of collective stabilization of steered particles in the plane is discussed in [12]. It is also of interest to study the discrete-time counterpart of the continuous time model (4). To this end we interpret (4) as follows: when $K < 0$ each agent moves towards the centroid p_θ, when $K > 0$ each agent moves away from the centroid p_θ. This interpretation suggests the discrete-time algorithm [10]

$$\theta_k[t+1] = \arg\left((1-\delta_k)e^{i\theta_k[t]} \pm \delta_k p_\theta[t]\right), \quad \delta_k \in (0,1), \quad k = 1, \ldots, N. \tag{6}$$

The update (6) amounts, for each particle, to moving towards the centroid (respectively away from it) in the complex plane and to project the result onto the manifold S^1 (see Fig.1). It is worth noting that (6) reduces to (4) as $\delta_k \to 0,\ k = 1, \ldots, N$.

The algorithms (6) and (4) make use of all-to-all communication to calculate the centroid p_θ that appears in the expression of the gradient (3). In a local information algorithm, this average quantity must be replaced by local information that might change over time. The next section summarizes important results on this topic, when the state space is an Euclidean space.

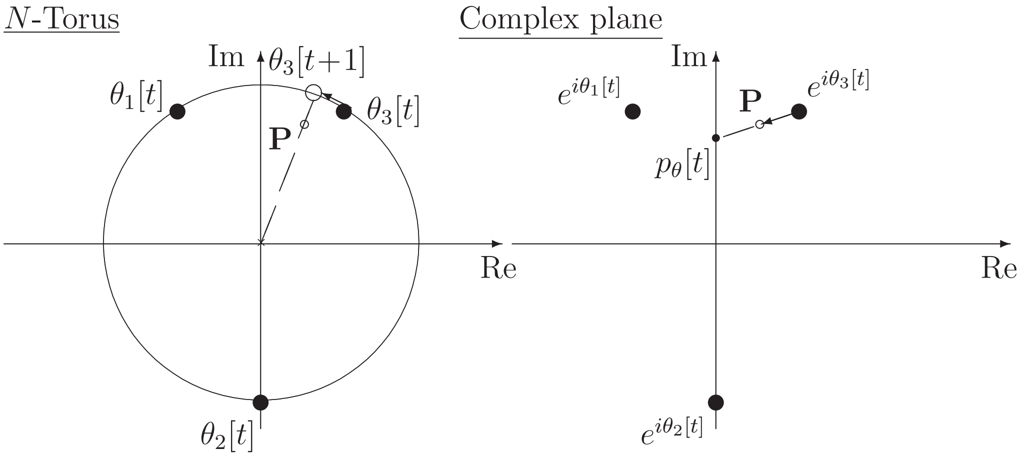

Fig. 1. Interpretation of (6) as a projection onto the manifold S^1 $(P \triangleq (1 - \delta_k)e^{i\theta_k[t]} + \delta_k p_\theta[t])$

3 Consensus in Euclidean space

In this section we recall some recent results about consensus algorithms in the Euclidean space. This *Consensus problem*, has received considerable attention in the recent years, see for instance [9,8,7,1].

Let $G = (\mathcal{V}, \mathcal{E}, A)$ be a weighted digraph (directed graph) where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and A is a weighted adjacency matrix with nonnegative elements a_{kj}. The node indices belong to the set of positive integers $\mathcal{I} \triangleq \{1, \ldots, N\}$. Assume that there are no self-cycles i.e. $a_{kk} = 0$, $\forall\, k \in \mathcal{I}$.

The graph Laplacian L associated to the graph G is defined as

$$L_{kj} = \begin{cases} \sum_i a_{ki},\ j = k \\ \\ -a_{kj}, \quad j \neq k. \end{cases}$$

The k-th row of L is defined by L_k. The in-degree (respectively out-degree) of node v_k is defined as $d_k^{in} = \sum_{j=1}^N a_{kj}$ (respectively $d_k^{out} = \sum_{j=1}^N a_{jk}$). The digraph G is said to be *balanced* if the in-degree and the out-degree of each node are equal, that is,

$$\sum_j a_{kj} = \sum_j a_{jk}, \qquad \forall\, i \in \mathcal{I}.$$

It is both of theoretical and practical interest to consider time-varying communication topologies. For example, in a network of moving agents, some of the existing links can fail and new links can appear when other agents enter an effective range of detection. In the following we assume that the communication topology is described by a time-varying graph $G(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$, where $A(t)$ is piece-wise continuous and bounded and $a_{kj}(t) \in \{0\} \cup [\beta, \gamma], \forall\, k, j,$ for some finite scalars $0 < \beta \leq \gamma$ and for all $t \geq 0$. The set of neighbors of node v_k at time t is denoted by $\mathcal{N}_k(t) \triangleq \{v_j \in \mathcal{V} : a_{kj}(t) \geq \beta\}$. We recall two definitions that characterize the concept of uniform connectivity for time-varying graphs.

Definition 1 *Consider a graph $G(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$. A node v_k is said to be connected to node v_j ($v_j \neq v_i$) in the interval $I = [t_a, t_b]$ if there is a path from v_k to v_j which respects the orientation of the edges for the directed graph $(\mathcal{N}, \cup_{t \in I} \mathcal{E}(t), \int_I A(\tau) d\tau)$.*

Definition 2 *$G(t)$ is said to be uniformly connected if there exists $T > 0$ such that for all t there is one node connected with all the other nodes across $[t, t+T]$.*

Consider a group of N agents with state $x_k \in X$, where X is an Euclidean space. The communication between the N-agents is defined by the graph G: each agent can sense only the neighboring agents, i.e. agent j receives information from agent i iff $i \in \mathcal{N}_j(t)$. We use the notation $k \sim j$ to indicate the presence of a communication link from agent j to agent k, i.e. $k \sim j$ iff $v_j \in \mathcal{N}_k$.

In continuous time, we consider the continuous dynamics

$$\dot{x}_k = \sum_{k\sim j} a_{kj}(t)(x_j - x_k), \qquad \forall k \in \mathcal{I}. \tag{7}$$

Using the Laplacian definition, (7) can be equivalently expressed as

$$\dot{x} = -L(t)\, x. \tag{8}$$

A discrete time version of (8) is

$$x[t+1] = x[t] - \varepsilon[t]L[t]x[t], \quad \varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, ..\varepsilon_N), \;\; \varepsilon_k \in (0, 1/d_k^{in}). \tag{9}$$

The bound on ε_k is connected to a centroid computation: for the value $\varepsilon_k = 1/d_k^{in}$, (9) becomes

$$x_k[t+1] = \frac{\sum_{k\sim j} a_{kj}[t]x_j[t]}{\sum_{k\sim j} a_{kj}[t]}.$$

Algorithms (8) and (9) have been widely studied in the literature and asymptotic convergence to a consensus value holds under mild assumptions on the communication topology. The following theorem summarizes some of the main results in [7], [8] and [9].

Theorem 2 *Let X be a finite-dimensional Euclidean space. Let $G(t)$ be a uniformly connected digraph and $L(t)$ the corresponding Laplacian matrix bounded and piecewise continuous in time. The solutions of (8) and (9) asymptotically converge to a consensus value $\alpha\mathbf{1}$ for some $\alpha \in X$. Furthermore if $G(t)$ is balanced for all t, and $\varepsilon_k = \varepsilon_j$ for all $j, k \in \mathcal{I}$, then $\alpha = \frac{1}{N}\sum_{i\in\mathcal{I}} x_i(0)$.* □

A general proof for Theorem 2 is based on the property that the convex hull of vectors $x_k \in X$ is non expanding along the solutions. For this reason, the assumption that X is an Euclidean space is essential (see e.g. [8]). Under the additional balancing assumption on $G(t)$, the norm $x^T x$ is non increasing. Moreover, the balancing assumption implies $\mathbf{1}^T L(t) = 0$, which implies that the average $\frac{1}{N}\sum_{j\in\mathcal{I}} x_j$ is an invariant quantity along the solutions.

To draw a connection between Section 2 and 3, it is of interest to rewrite Algorithm (8) in the particular case of a *complete* graph, i.e. $L = N\Pi$, with $\Pi = I - \frac{\mathbf{1}\mathbf{1}^T}{N}$ a projector. Then (8) rewrites as

$$\dot{x}_k = -(x_k - \frac{1}{N}\sum_{j\in\mathcal{I}} x_j) = -\frac{1}{2}\frac{\partial}{\partial x_k} < x, \Pi\, x > \tag{10}$$

yielding the interpretation of (10) as a descent algorithm for the cost function $\|\Pi\, x\|^2 = \frac{1}{N} < x, Lx >$ in a way analogous to the algorithm (4) on the torus. For this reason, the consensus algorithm (8) can be viewed as a local information algorithm that retains the convergence property of the global information descent algorithm (10).

4 Synchronization and Balancing on the N-Torus: static algorithms

We now return to the problem of designing local information algorithms to optimize the cost function $V(\theta)$. In light of the results of the preceding section, the main idea is to replace the global quantity p_θ with a local one in the dynamics (4) and (6). This is the approach followed in [14] and [3], to generalize (4) to arbitrary communication topologies. In continuous-time, (4) is replaced by

$$\dot{\theta}_k = K < L_k(t)e^{i\theta}, ie^{i\theta_k} >= -K \sum_{k\sim l} a_{kl}(t)\sin(\theta_l - \theta_k), \qquad \forall\, k \in \mathcal{I}, \qquad (11)$$

where $e^{i\theta} = [e^{i\theta_1}, \ldots, e^{i\theta_N}]^T$.

The discrete time counterpart proposed in [10] is

$$\theta_k[t+1] = \arg\left((1-\delta_k)e^{i\theta_k[t]} \pm \delta_k L_k[t]\, e^{i\theta[t]}\right), \quad \delta_k \in (0,1), \quad \forall\, k \in \mathcal{I}. \quad (12)$$

We note that the dynamics (12) particularize to the Vicsek model when $\delta_k = 1/d_k^{in}$. This model was proposed in [17] to describe the discrete-time evolution of interacting particles that move with unit velocity in the plane.

The dynamics (11) and (12) should be viewed as the counterpart of the dynamics (8) and (9) on the N-torus. In particular, (11) and (12) linearize to (8) and (9) in the neighborhood of a synchronized state. Nevertheless, the convergence theory of (11) and (12) is less complete than the convergence theory summarized in the previous section. If the graph is undirected and fixed, then (11) is a gradient algorithm for the Laplacian potential $V = \frac{1}{2} < e^{i\theta}, L\, e^{i\theta} >$ and solutions converge to the critical points of V [14]. The synchronized state is always a (global) minimum of V but the potential may posses other local minima, in which case the convergence to a consensus value does not hold globally.

The discrete algorithm (12) is also a descent (or ascent) algorithm for the Laplacian potential, provided that the states are updated asynchronously or provided that δ_k is small enough [10]. For time-varying and directed graphs, that is under the general assumption of Theorem 2 on $G(t)$, it is an open question whether convergence to a consensus value is generic. Only local results have been proposed in the literature [8,3]. As suggested in [8], the proof argument of Theorem 2 can be extended to (11) and (12) by mapping the dynamics onto the Euclidean space, but this requires to restrict the set of critical conditions to half a circle. The lack of global convergence results for the descent algorithm (11) and (12) leads us to propose a dynamic algorithm in the next sections.

The simple idea behind the proposed approach is to combine the gradient system defined on the N-Torus (Section 2) and the consensus algorithm defined in $\mathbb{C}^N$ (Section 3). Following the lines of [11], the local information provided by the consensus algorithm is used to estimate the global information required by the gradient algorithm.

5 Global synchronization on the N-Torus: dynamic algorithms

First we consider the synchronization problem. For notational convenience we use the following conventions. We denote by θ_{jk} the difference between the angles θ_j and θ_k, i.e. $\theta_{jk} := \theta_j - \theta_k$. We denote by $\Delta\theta[t+1]$ the increment of angle θ at time $t+1$, i.e. $\Delta\theta[t+1] := \theta[t+1] - \theta[t]$. Synchronized states coincide with the global maxima of the cost function $V = \frac{N}{2} \mid p_\theta \mid^2$. We seek to replace the (*global* information) gradient algorithm

$$\dot{\theta}_k = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_{jk}) =< p_\theta, ie^{i\theta_k} >,$$

by the *local* information algorithm

$$\begin{cases} \dot{\theta}_k = < x_k, ie^{i\theta_k} >, \\ \dot{x}_k = -L_k(t)x, \quad k \in \mathcal{I}, \quad x_k \in \mathbb{C}, \end{cases} \tag{13}$$

and

$$\theta_k[t+1] = \arg\left(\delta_k p_\theta[t] + (1-\delta_k)e^{i\theta_k[t]}\right), \quad \delta_k \in (0,1), \quad \forall\, k \in \mathcal{I}.$$

by

$$\begin{cases} \theta_k[t+1] = \arg\left((1-\delta_k)e^{i\theta_k[t]} + \delta_k \frac{x_k[t]}{\|x_k[t]\|}\right), & \delta_k \in (0,1) \\ x[t+1] = x[t] - \varepsilon[t]L[t]x[t], & \varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, ..\varepsilon_N) \end{cases} \tag{14}$$

where $\varepsilon_k \in (0, 1/d_k^{in})$ and $x_k[t] \in \mathbb{C}$. The convergence analysis of algorithms (13) and (14) is straightforward. Since the consensus algorithm is decoupled from the optimization algorithm, Theorem 2 guarantees that each local estimate x_k converges to a consensus value $\alpha =: \mid \alpha \mid e^{i\phi}$. As a consequence, the first equation of (13) and (14) asymptotically converge to a system whose only stable equilibrium is the synchronized state. Before detailing the convergence analysis, we express algorithms (13) and (14) in shape coordinates in order to recover the invariance of the phase dynamics to rigid rotations. Defining

$$r_k = (x_k)e^{-i\,\theta_k}, \qquad \forall\, k \in \mathcal{I},$$

(13) is rewritten as

$$\begin{cases} \dot{\theta}_k = < r_k, i >, & \forall\, k \in \mathcal{I} \\ \dot{r}_k = -ir_k\dot{\theta}_k - \sum_{j=1}^N L_{kj}(t)\, r_j\, e^{i\theta_{jk}}, & r_k \in \mathbb{C} \end{cases} \tag{15}$$

In the same way we rewrite the discrete-time algorithm (14) as

$$\begin{cases} \Delta\theta_k[t{+}1] = \arg\left(\delta_k \frac{r_k[t]}{\|r_k[t]\|} + (1-\delta_k)\right) \\ r_k[t+1] = \left(r_k[t] - \varepsilon[t]\sum_{j=1}^N L_{kj}[t] r_j[t]\, e^{i\theta_{jk}[t]}\right)\, e^{-i\Delta\theta_k[t+1]}, \quad r_k \in \mathbb{C}. \end{cases} \tag{16}$$

Theorem 3 *Suppose that the communication graph $G(t)$ is uniformly connected and that $L(t)$ is bounded and piecewise continuous. Then all the solutions of the decentralized algorithms (15) and (16) asymptotically converge to an equilibrium. Moreover, the only stable equilibrium in the shape space T^N/S^1 is the synchronized state characterized by N identical phases. Furthermore, if $G(t)$ is balanced for all t, $\varepsilon_k = \varepsilon_j$ for all $j, k \in \mathcal{I}$ and $r_k(0) = 1$, for all $k \in \mathcal{I}$, then the asymptotic consensus value for $e^{i\theta_k}$ is $\alpha = (\frac{1}{N}\sum_{i\in\mathcal{I}} e^{i\theta_i(0)})$, that is the centroid $p_\theta(0)$ of the initial condition.* □

Proof: (continuous time) Set $x_k = r_k\, e^{i\theta_k}$. Then $x(t)$ obeys the consensus dynamics $\dot{x} = -L(t)x$, which implies that the solutions converge to a consensus value $\alpha =: \mid \alpha \mid e^{i\phi}$. This implies that the dynamics

$$\dot{\theta}_k =< r_k, i >, \tag{17}$$

asymptotically converge to the (time-invariant) dynamics

$$\dot{\theta}_k = < \alpha, ie^{i\theta_k} > = | \alpha | \sin(\phi - \theta_k), \tag{18}$$

for $\forall\, k \in \mathcal{I}$. Since the consensus dynamics for $x(t)$ are invariant with respect to translations in the plane, for any particular graph sequence, α has an equal probability to take any value in the complex plane if the initial conditions $x_k(0)$ are randomly chosen (in the complex plane). This is sufficient to conclude that $\alpha \neq 0$ with probability 1. Solutions of the complete system (15) are known to converge to a chain recurrent set of the limiting (autonomous) system (18) [18]. The limiting system is decoupled into N identical scalar systems whose only chain recurrent sets are the two equilibria of (18) (one stable node and one unstable node). Then the only limit sets of the local information algorithm (15) are equilibria that satisfy $\theta_k = \phi \operatorname{mod} \pi$ for all k. The synchronized equilibrium $\theta = \mathbf{1}\phi$ is exponentially stable while all other equilibria are exponentially unstable. If $G(t)$ is balanced, it follows from Theorem 2 that $\alpha = \frac{1}{N}\sum_{i\in\mathcal{I}} r_i(0)e^{i\theta_i(0)} = p_\theta(0)$.
The proof of the discrete time counterpart follows the same lines and is omitted. ■

We conclude that synchronization on the circle can be achieved with a local information algorithm whose exchanged information is not only the relative phase but also the estimate of a vector that serves as a consensus reference direction. The global convergence analysis obtained in this way is in contrast with the local convergence analysis proposed in [8,3], for the algorithms (11) and (12). The numerical simulation in Fig.2 illustrates a situation where the (dynamic) local information algorithm (13) achieves synchronization while the (static) algorithm (11) fails to converge. In this example, the communication is a fixed ring topology and the initial phase distribution spreads over more than half a circle. The static algorithm (11) converges to a balanced state where $p_\theta = 0$, which is a local minimum of the potential $\frac{1}{2} < e^{i\theta}, L\, e^{i\theta} >$.

6 Global balancing on the N-Torus: dynamic algorithms

Balanced states coincide with the global minima of the cost function $V = \frac{N}{2} \mid p_\theta \mid^2$. Similarly to the synchronization algorithms discussed in the

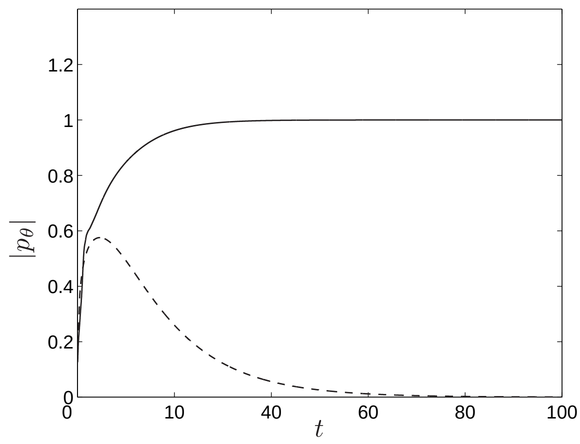

Fig. 2. Comparison of the behavior of the synchronization parameter $|p_\theta|$ for two different local information algorithms when the initial conditions spread over the entire circle: the (dynamic) algorithm (15)(full line); the (static) algorithm (11) (dash line). Only the first algorithm achieves synchronization. The simulation involves $N = 20$ particles with a random initial condition.

previous section, we now seek to replace the (global information) gradient algorithm with a local one. To this end, we consider the continuous-time system

$$\begin{cases} \dot{\theta}_k = - < r_k, i >, \\ \dot{r}_k = -i(r_k - 1)\dot{\theta}_k - \sum_{j=1}^{N} L_{kj}(t)\, r_j\, e^{i\theta_{jk}}, \end{cases} \tag{19}$$

where $r_k(0) = 1$, $\forall\, k \in \mathcal{I}$, and its discrete-time version

$$\begin{cases} \Delta\theta_k[t{+}1] = \arg\Big((1-\delta_k) - \delta_k r_k[t+1] e^{i\Delta\theta_k[t+1]}\Big) \\ r_k[t+1] = 1 + \Big(r_k[t] - 1 - \varepsilon[t] \sum_{j=1}^{N} L_{kj}[t] r_j[t]\, e^{i\theta_{jk}[t]}\Big)\; e^{-i\Delta\theta_k[t+1]}, \end{cases} \tag{20}$$

where $r_k(0) = 1$, $\forall\, k \in \mathcal{I}$, and $\varepsilon \in (0, \frac{1}{d_{\max}})$, $d_{\max} = \max_{k\in\mathcal{I}} d_k^{in}$.

Theorem 4 *Suppose that the communication graph $G(t)$ is uniformly connected and balanced for all $t \geq 0$ and that $L(t)$ is bounded and piecewise continuous. Then all the solutions of the decentralized algorithms (19) and (20) asymptotically converge to an equilibrium. Moreover, the only stable limit set is the set of balanced states characterized by $p_\theta = 0$.* □

Proof: (continuous time) Set $x_k = r_k e^{i\theta_k}$. The solution $x(t)$ satisfies the dynamics

$$\dot{x} = -L(t)x + \frac{d}{dt}e^{i\theta}. \tag{21}$$

The Lyapunov function

$$W(x) = \frac{1}{2} < x, x >,$$

is not increasing along the solutions of (19): note that, since the graph is balanced, $L(t)$ is a positive semi-definite matrix [19] and then

$$\dot{W} = - < L(t)x, x > - \sum_{k=1}^{N} < x_k, ie^{i\theta_k} >^2 = - < L(t)x, x > - \sum_{k=1}^{N} \dot{\theta}_k^2 \leq 0. \tag{22}$$

We deduce from (22) that $\dot{\theta}$ is a function in $L_2(0, \infty)$ since (22) implies that

$$\lim_{t\to\infty} \int_0^t \sum_{k=1}^{N} \dot{\theta}_k^2(\tau) d\tau = W(x(0)) - \lim_{t\to\infty} \left(W(x(t)) - \int_0^t < L(\tau)x(\tau), x(\tau) > d\tau \right) \leq \frac{N}{2}.$$

We deduce from (22) that $\dot{\theta}$ is a function in $L_2(0, \infty)$ and that x is uniformly bounded. To prove that $\dot{\theta}$ asymptotically converges to zero observe that

$$\ddot{\theta}_k = < L_k(t)\, x, ie^{i\theta_k} > + (< x_k, e^{i\theta_k} > -1)\dot{\theta}_k$$

is uniformly bounded, which implies that $\dot{\theta}$ is Lipschitz continuous. We conclude that $\dot{\theta}$ is uniformly continuous. Then $\dot{\theta}$ is a uniformly continuous function in $L_2(0, \infty)$ and from Barbalat's Lemma we obtain that $\dot{\theta} \to 0$ as $t \to \infty$ [20].

Thanks to the balancing assumption on the graph, $\mathbf{1}$ is a left eigenvector of $L(t)$, and we obtain from (21) that

$$\frac{1}{N} < \mathbf{1}, \dot{x} > = \frac{1}{N} < \mathbf{1}, \frac{d}{dt}e^{i\theta} > . \tag{23}$$

Integrating both sides of (23), and using the fact that $x_k(0) = e^{i\theta_k(0)}$, one concludes that $\frac{1}{N}\sum_{i\in\mathcal{I}} x_i(t) = p_\theta$ for all $t \geq 0$. Because $x(t)$ converges to a consensus equilibrium, each component x_k must asymptotically converge to p_θ. As a consequence, the dynamics $\dot{\theta}_k = - < r_k, i >$ asymptotically converge to the time-invariant dynamics

$$\dot{\theta}_k = - < p_\theta, ie^{i\theta_k} >, \qquad \forall\, k \in \mathcal{I}. \tag{24}$$

Since $\dot{\theta}$ is asymptotically convergent to zero, the solutions asymptotically converge to a set of equilibria of (24). We conclude that $\theta(t)$ asymptotically converges to the critical set of V and, form Theorem 1, that only the set of balanced states is asymptotically stable.

(discrete time) Set $x_k = r_k e^{i\theta_k}$. The solution $x[t]$ satisfies the dynamics

$$x[t+1] = x[t] - \varepsilon[t]L[t]x[t] + e^{i\theta_k[t+1]} - e^{i\theta_k[t]}. \tag{25}$$

As in continuous-time consider the Lyapunov function $W(x) = \frac{1}{2} < x, x >$.

First, we note that, because $(I - \varepsilon L[t])$ is a doubly stochastic matrix [2], then

$$\|I - \varepsilon[t]L[t]x[t]\|^2 \leq \|x[t]\|^2$$

so that

$$W\left(x[t+1]\right) - W(x[t]) \leq \|x[t+1]\|^2 - \|I - \varepsilon[t]L[t]x[t]\|^2. \tag{26}$$

Next, we observe that

$$\begin{aligned}
&\|x[t+1]\|^2 - \|I - \varepsilon[t]L[t])x[t]\|^2 \\
&=< x[t+1], e^{i\theta[t+1]} - e^{i\theta[t]} > + < e^{i\theta[t+1]} - e^{i\theta[t]}, (I - \varepsilon[t]L[t])x[t] > \\
&= 2 < x[t+1], e^{i\theta[t+1]} - e^{i\theta[t]} > -\|e^{i\theta[t+1]} - e^{i\theta[t]}\|^2 \\
&\leq -\|e^{i\theta[t+1]} - e^{i\theta[t]}\|^2,
\end{aligned} \tag{27}$$

where the last inequality uses the property that, by definition of $\theta_k[t+1]$,

$$< x_k[t+1], e^{i\theta_k[t+1]} > \leq < x_k[t+1], e^{i\theta_k[t]} >,$$

for every k. Using (26) and (27) and summing over t yields

$$\sum_{t=0}^{\infty} \|e^{i\theta[t+1]} - e^{i\theta[t]}\|^2 \leq W(x[0]).$$

The rest of the proof follows from the argument used in continuous-time. ■

It is worth noting that in contrast to the algorithms (15) and (16), algorithms (19) and (20) are coupled; moreover, this coupling leads (in the discrete-time version) to implicit update equations through the presence of $r_k[t+1]$ in the nonlinear update equation for θ_k.

Theorems 3 and 4 generalize the global convergence results of the all-to-all gradient control (4) and (6) under mild assumptions on the communication graph. This generalization is obtained at the prize of increased communication between the communicating agents. They must communicate not only their relative configuration variables θ_{jk} but also their estimates r_k and r_j. In both theorems, the variable r_k can be interpreted as a local estimate of p_θ in the local frame attached to particle k while x_k is the local estimate in the absolute (reference) frame. In design applications, it might be meaningful to exchange additional information between communicating agents in order to relax the cost of global communication architectures.

7 Conclusion

In this paper a novel algorithm is proposed for synchronization and balancing in phase models on the N-torus. In the spirit of earlier work on phase synchronization [4], we view synchronization as the task of maximizing the norm of the centroid and balancing as the task of minimizing the norm of the centroid. Gradient-based algorithms require global information because the update law of each agent requires the centroid information. In the proposed algorithm, this global information is estimated on the basis of locally available information, in such a way that the global convergence properties of the original algorithm are asymptotically recovered by the new one. The global convergence analysis on the N-torus is a distinctive feature of the present work with respect to previous convergence results that have focused on decentralized consensus algorithms in the Euclidean space. The proposed approach extends beyond phase models on the N-torus. In particular, it can be used to extend in a local information framework global information algorithms proposed in [12], see [13].

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