EXTENSIONS AND RESTRICTIONS OF WYTHOFF'S GAME PRESERVING WYTHOFF'S SEQUENCE AS SET OF $\mathcal{P}$ POSITIONS

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http://www.discmath.ulg.ac.be/

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http://hdl.handle.net/2268/100440

LIAFA, Paris, October 21th, 2011

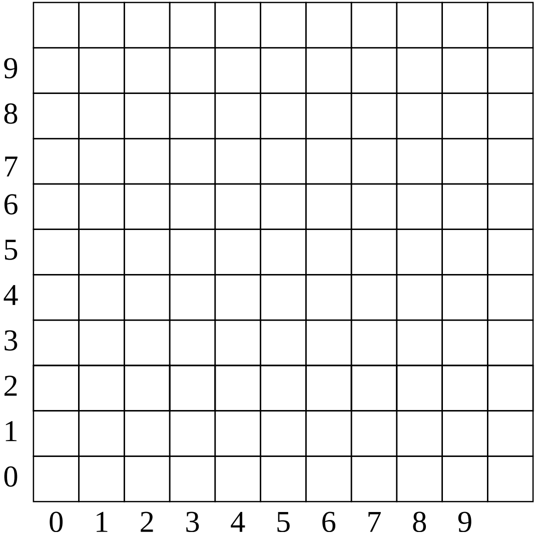
W. A. Wythoff, A modification of the game of Nim,
Nieuw Arch. Wisk. **7** (1907), 199–202.

RULES OF THE GAME

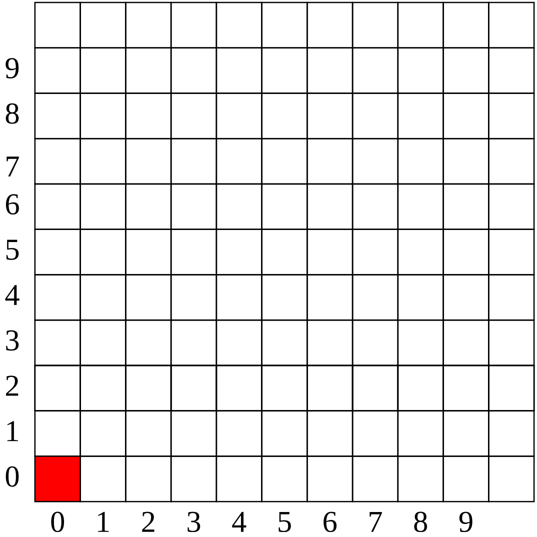
- Two players play alternatively
- Two piles of tokens
- Remove
  - any positive number of tokens from one pile or,
  - the same positive number from the two piles.
- The one who takes the last token wins the game (last move wins).

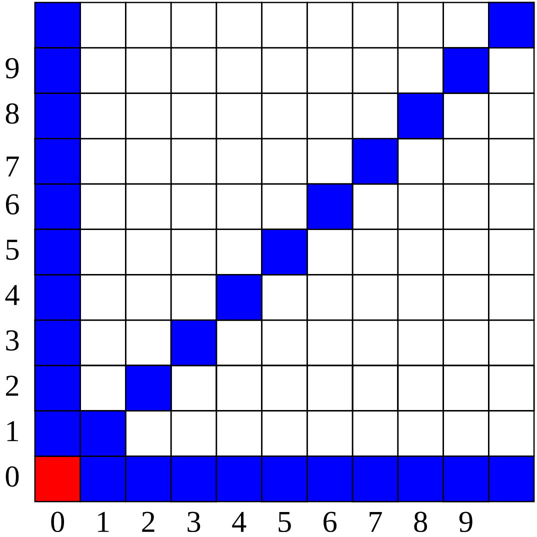
Set of moves : $\{(i,0),\ i>0\}\cup\{(0,j),\ j>0\}\cup\{(k,k),\ k>0\}$

Wythoff's game or "catching the queen"

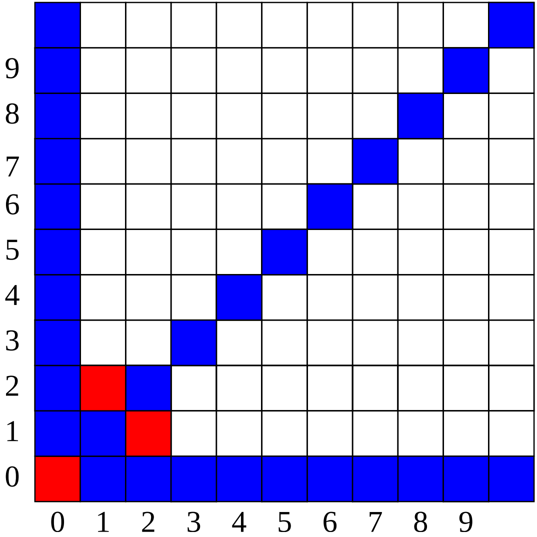


WYTHOFF'S GAME OR "CATCHING THE QUEEN"




9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9

WYTHOFF'S GAME OR "CATCHING THE QUEEN"



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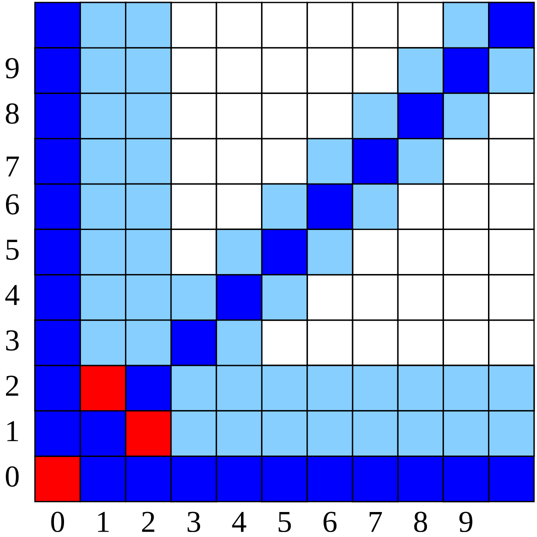

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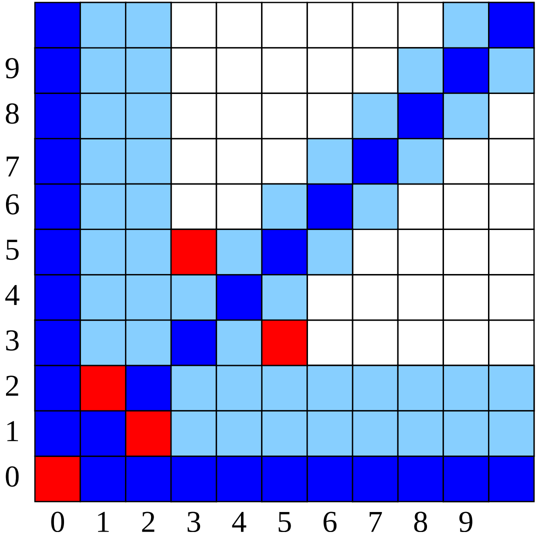

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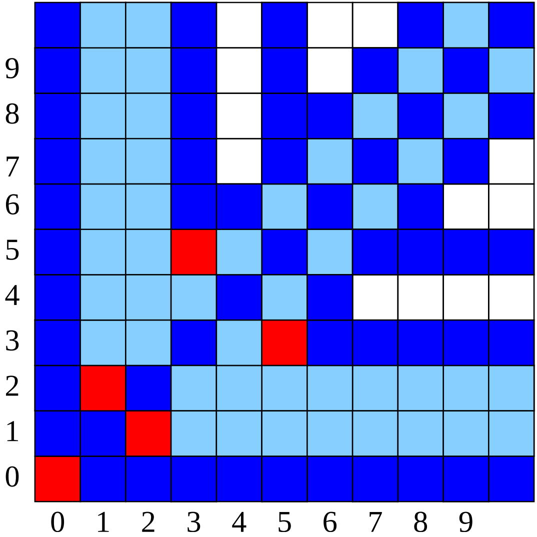

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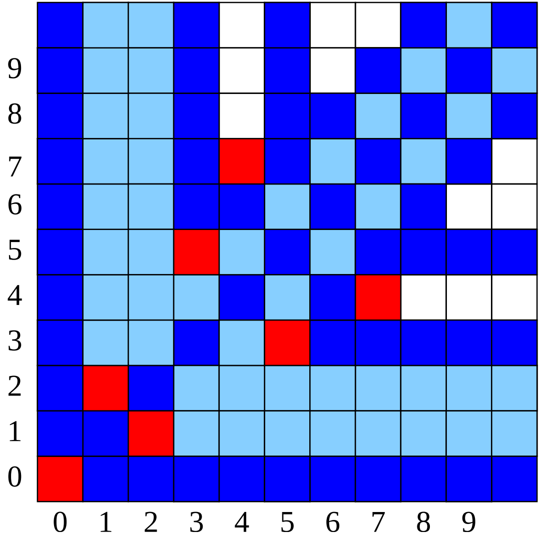

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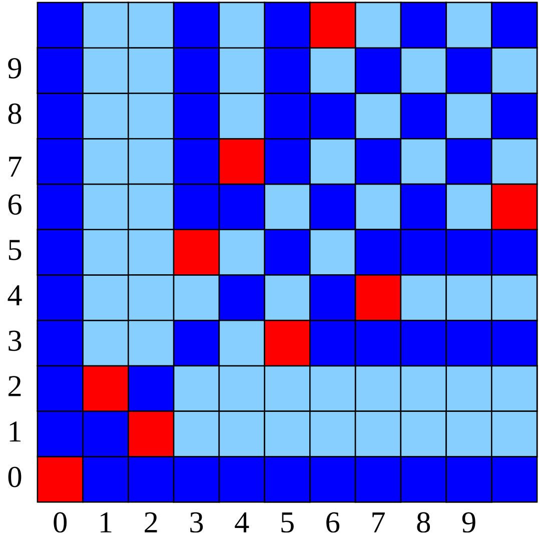
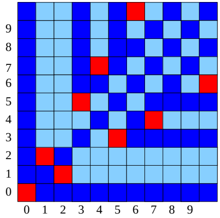




$(0,0),\ (1,2),\ (3,5),\ (4,7),\ (6,10),\ \ldots$

P-POSITION

A $\mathcal{P}$-position is a position q from which the *previous* player (moving to q) can force a win.

N-POSITION

A $\mathcal{N}$-position is a position p from which the *actual* player has an option leading ultimately to win the game.

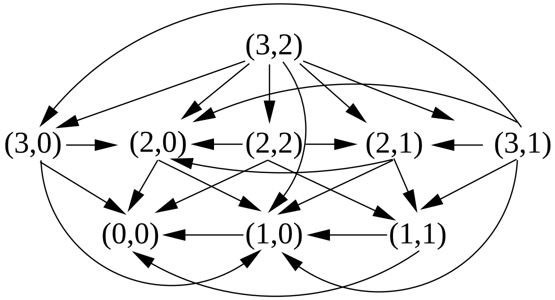
Question : **Are all positions $\mathcal{N}$ or $\mathcal{P}$?**

Game graph

Initial position (i_0, j_0), by symmetry, take only $(i \geq j)$

- **Vertices** : $\{(i,j),\ i \leq i_0, j \leq j_0\}$
- **Edges** : from each position to all its options :

$$\begin{array}{l|lcl|l} i > 0 & (i,j) & \rightarrow & (i-k,j) & k = 1,\ldots,i \\ j > 0 & (i,j) & \rightarrow & (i,j-k) & k = 1,\ldots,j \\ i,j > 0 & (i,j) & \rightarrow & (i-k,j-k) & k = 1,\ldots,\min(i,j) \end{array}$$



Game graph

Remark

Due to the rules, the game graph for Wythoff's game is **acyclic**.

Theorem [Berge]

Any finite acyclic digraph has a unique kernel.

Moreover, this kernel can be obtained efficiently.

Reminder/Definition of a kernel

A kernel in a graph $G = (V, E)$ is a subset $W \subseteq V$

- stable : $\forall x, y \in W$, $(x, y) \notin E$
- absorbing : $\forall x \in V \setminus W$, $\exists y \in W : (x, y) \in E$.

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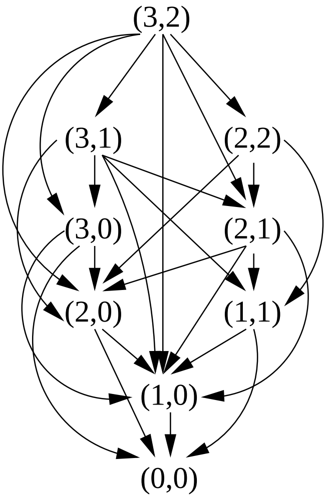
Game graph

Bottom-Up approach from the sinks
(they belong to the kernel because it is absorbing)



GAME GRAPH

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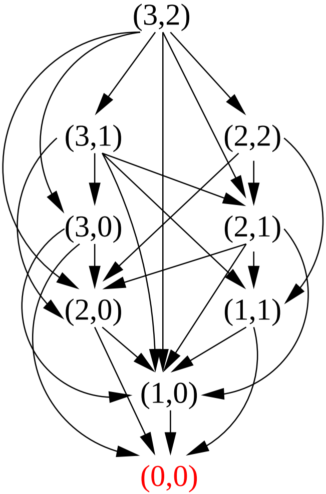

Game graph

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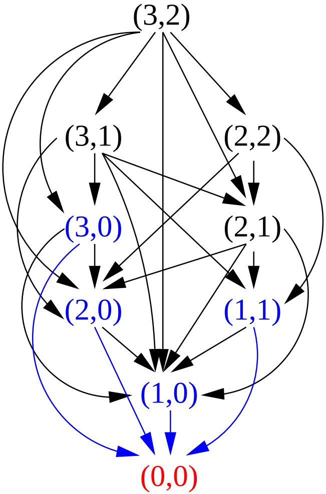

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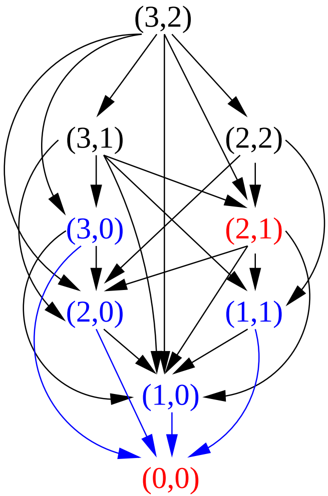

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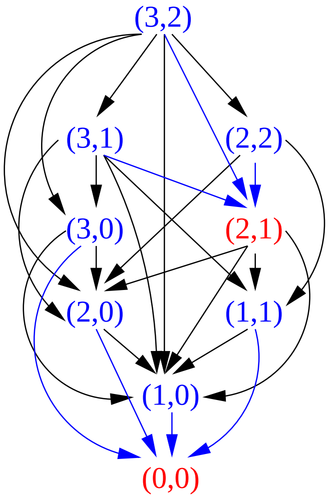

Game graph

For Wythoff's game, its game graph has a unique kernel K.

- stable : from a position in K, you always play out of K,
- absorbing : from a position outside K, you can play into K,
- $(0,0)$ has to belong to K, otherwise K won't be absorbing.

Corollary (for any impartial acyclic game)

The set of $\mathcal{P}$-positions is exactly the kernel K
and all the other positions are $\mathcal{N}$-positions.

$\{\mathcal{P}\text{-positions}\} \supseteq K$

If p is a position in K, then it is a $\mathcal{P}$-position
because there is a *winning strategy* outside K.

$\{\mathcal{P}\text{-positions}\} \subseteq K$

If p is a $\mathcal{P}$-position not in K, then there is a move from p to K,
thus p is a $\mathcal{N}$-position !

GAME GRAPH

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If p is a $\mathcal{P}$-position not in K, then there is a move from p to K,
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A USUAL PROOF TECHNIQUE

To prove that a given set S of positions is the set of $\mathcal{P}$-positions of a game, one shows that S is stable and absorbing with respect the game moves.

P-POSITION OF THE WYTHOFF'S GAME I

$(A_n, B_n)_{n \geq 0} = (0,0),\ (1,2),\ (3,5),\ (4,7),\ \ldots$

$$\forall n \geq 0, \quad \begin{cases} A_n = \mathit{Mex}\{A_i, B_i \mid i < n\} \\ B_n = A_n + n \end{cases}$$

P-POSITION OF THE WYTHOFF'S GAME II

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$
$\mathcal{F}$	a	b	a	a	b	a	b	a	a	b	a	a	b	a	

P-POSITIONS OF THE WYTHOFF'S GAME III

$$(A_n, B_n)_{n \geq 0} = (\lfloor n\,\tau \rfloor,\ \lfloor n\,\tau^2 \rfloor).$$

- A.S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly* **89** (1982), 353–361.
- A.S. Fraenkel, Heap games, Numeration systems and Sequences, *Annals of Combinatorics* **2** (1998), 197–210.
- A.S. Fraenkel, The Raleigh Game, INTEGERS (2007).
- E. Duchêne, M.R., A morphic approach to combinatorial games: the Tribonacci case, *RAIRO Theoret. Inform. Appl.* **42** (2008), 375–393.
- E. Duchêne, M.R., A class a cubic Pisot unit games, *Monat. für Math.* **155** (2008), 217–249.

Different sets of moves / more piles

$$\downarrow$$

Different sets of $\mathcal{P}$-positions to characterize...

Our goal / Dual question

Consider invariant extensions or restrictions of Wythoff's game that keep the set of $\mathcal{P}$-positions of Wythoff's game unchanged.

Characterize the different sets of moves...

$$\downarrow$$

Same set of $\mathcal{P}$-positions as Wythoff's game

Definition, E. Duchêne, M. R., TCS 411 (2010)

A removal game G is invariant, if for all positions $p = (p_1, \ldots, p_\ell)$ and $q = (q_1, \ldots, q_\ell)$ and any move $x = (x_1, \ldots, x_\ell)$ such that $x \preceq p$ and $x \preceq q$ then, the move $p \to p - x$ is allowed if and only if the move $q \to q - x$ is allowed.

- Nim or Wythoff game are invariant games
- Raleigh game, the Rat and the Mouse game, Tribonacci game, Cubic Pisot games, ... are NOT invariant

Non-invariant game

Remove an odd number of tokens from a position (a, b) if a or b is a prime number, and an even number of tokens otherwise.

Very recently, Nhan Bao Ho (La Trobe Univ., Melbourne), *Two variants of Wythoff's game preserving its $\mathcal{P}$-positions*:

- A restriction of Wythoff's game in which if the two entrees are not equal then removing tokens from the smaller pile is not allowed.
- An extension of Wythoff's game obtained by adjoining a move allowing players to remove k tokens from the smaller pile and ℓ tokens from the other pile provided $\ell < k$.

Our goal / Dual question

Consider invariant extensions or restrictions of Wythoff's game that keep the set of $\mathcal{P}$-positions of Wythoff's game unchanged.

- We characterize all moves that can be adjoined while preserving the original set of $\mathcal{P}$-positions.
- Testing if a move belong to such an extended set of rules can be done in polynomial time.

During our journey...

Canonical construction [Cobham'72]

Let $k \geq 2$. A sequence $x = (x_n)_{n\geq 0} \in A^{\mathbb{N}}$ is k-automatic IFF it is the image under a coding of an infinite word generated by a prolongable k-uniform morphism.

Example

Characteristic sequence of $\{n \mid \exists i, j \geq 0 : n = 2^i + 2^j\} \cup \{1\}$

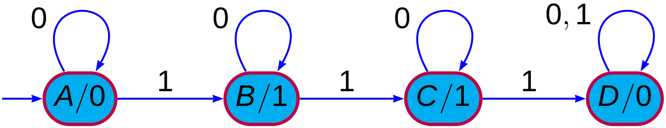
$$g : \begin{cases} A & \mapsto & AB \\ B & \mapsto & BC \\ C & \mapsto & CD \\ D & \mapsto & DD \end{cases} \qquad f : \begin{cases} A & \mapsto & 0 \\ B & \mapsto & 1 \\ C & \mapsto & 1 \\ D & \mapsto & 0 \end{cases}$$

$$g^\omega(A) = ABBCBCCDBCCDCDDDBCCDCDDDCDDDDDDD\cdots$$

$$f(g^\omega(A)) = 01111110111010001110100010000000\cdots$$

During our journey...

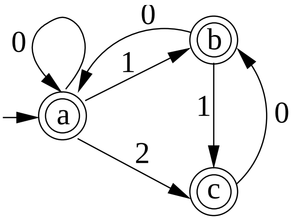
$$f(g^{\omega}(A)) = 01111110111010001110100010000000\cdots$$



$$x_n = \tau(q_0 \cdot \mathrm{rep}_2(n)).$$

Canonical construction: (non-uniform) morphisms $\to$ automata

$$\varphi : a \mapsto abc,\ b \mapsto ac,\ c \mapsto b$$

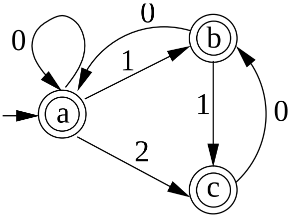


$$\varphi^\omega(a) = abcacbabcbacabcacbacabcbabcacb\cdots$$

Consider the language $L = L(\mathcal{M}) \setminus 0\{0,1,2\}^*$.

Remark: Positions in $\varphi^\omega(a)$ are counted from **1**.

Take the words of L with radix order (abstract system)



n	w_n			n	w_n		
0	ε	a	1	10	200	a	11
1	1	b	2	11	201	c	12
2	2	c	3	12	1000	a	13
3	10	**a**	**4**	13	1001	b	14
4	11	c	5	14	1002	c	15
5	20	b	6	15	1010	a	16
6	100	a	7	16	1011	c	17
7	101	b	8	17	1020	b	18
8	102	c	9	18	1100	a	19
9	110	b	10	19	1101	c	20

Not a “positional” system, no sequence behind.

EXAMPLE :

The 4th letter is a, it corresponds to $w_3 = 10$.

Since $\varphi(a) = abc$, we consider $\begin{cases} w_3 0 = 100 = w_i \\ w_3 1 = 101 = w_{i+1} \\ w_3 2 = 102 = w_{i+2} \end{cases}$

then the $(i+1)$st, $(i+2)$st, $(i+3)$st letters are a, b, c.

$$\text{rep}_L(i) := w_i, \qquad \text{val}_L(w_i) := i$$

Proposition

Let the nth letter of $\varphi^\omega(a)$ be σ and w_{n-1} be the nth word in L. If $\varphi(\sigma) = x_1 \cdots x_r$, then $x_1 \cdots x_r$ appears in $\varphi^\omega(a)$ in positions $\text{val}_L(w_{n-1}0)+1, \ldots, \text{val}_L(w_{n-1}(r-1))+1$.

For Wythoff's game: Fibonacci word $\mathcal{F}$, $L = 1\{01, 0\}^* \cup \{\varepsilon\}$ and we get the usual Fibonacci system $\rho_F : \mathbb{N} \to L$, $\pi_F : L \to \mathbb{N}$.

Corollary

- If the nth letter in $\mathcal{F}$ is a ($n \geq 1$), then this a produces through φ a factor ab occupying positions $\pi_F(\rho_F(n-1)0)+1$ and $\pi_F(\rho_F(n-1)1)+1$.
- If the nth letter in $\mathcal{F}$ is b ($n \geq 1$), then this b produces through φ a letter a occupying position $\pi_F(\rho_F(n-1)0)+1$.

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PROPOSITION

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- If the nth letter in $\mathcal{F}$ is b ($n \geq 1$), then this b produces through φ a letter a occupying position $\pi_F(\rho_F(n-1)0) + 1$.

Reminder on Fibonacci numeration system

Fibonacci sequence : $F_{i+2} = F_{i+1} + F_i$, $F_0 = 1$, $F_1 = 2$

Use greedy expansion, $\ldots, 21, 13, 8, 5, 3, 2, 1$

n	$\rho_F(n)$	n	$\rho_F(n)$	n	$\rho_F(n)$
1	1	8	10000	15	100010
2	10	9	10001	16	100100
3	100	10	10010	17	100101
4	101	11	10100	18	101000
5	1000	12	10101	19	101001
6	1001	13	100000	20	101010
7	1010	14	100001	21	1000000

E. Zeckendorf, Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liège* **41** (1972), 179–182.

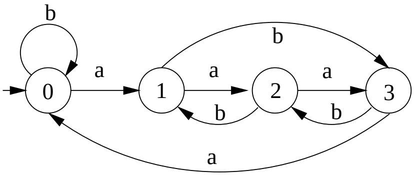
In fact, this is a special case of the following result.

THEOREM [A. MAES, M.R. '02]

The set of S-automatic sequences is exactly the set of morphic words.

Take any regular language with radix order $\oplus$ DFAO

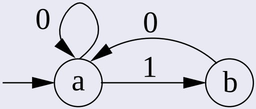
i	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\text{rep}_S(i)$	ε	a	b	aa	ab	bb	aaa	aab	abb	bbb	$\cdots$



$0102303120023101012302303120312023100231012301012 3\cdots$

n	1	2	3	4	5	6	7	8	9	10	11	12
	a	b	a	a	b	a	b	a	a	b	a	a
A_i	1		3	4		6		8	9		11	12
B_i		2			5		7			10		
$\rho_F(n-1)$	ε	1	10	100	101	1000	1001	1010	10000	10001	10010	10100

P-positions of the Wythoff's game IV



For all $n \geq 1$, we have

$$
\begin{aligned}
A_n &= \pi_F(\rho_F(n-1)0) + 1 \\
B_n &= \pi_F(\rho_F(A_n-1)1) + 1.
\end{aligned}
$$

More ?

Can we get a “morphic characterization” of the Wythoff’s matrix ?

$$(P_{i,j})_{i,j\geq 0} = \begin{array}{|cccccccccccc}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & & & & & & \ddots
\end{array}$$

Let's try something...

$$\varphi : a \mapsto \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad b \mapsto \begin{array}{|c|}\hline i \\ \hline e \\ \hline\end{array} \quad c \mapsto \begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \quad d \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad e \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline\end{array}$$

$$f \mapsto \begin{array}{|c|c|}\hline g & b \\ \hline h & d \\ \hline\end{array} \quad g \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline h & d \\ \hline\end{array} \quad h \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline\end{array} \quad i \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline h & d \\ \hline\end{array}$$

$$j \mapsto \begin{array}{|c|}\hline k \\ \hline c \\ \hline\end{array} \quad k \mapsto \begin{array}{|c|c|}\hline l & m \\ \hline c & d \\ \hline\end{array} \quad l \mapsto \begin{array}{|c|c|}\hline k & m \\ \hline c & d \\ \hline\end{array} \quad m \mapsto \begin{array}{|c|}\hline i \\ \hline h \\ \hline\end{array}$$

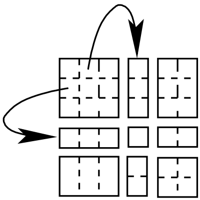
and the coding

$$\mu : e, g, j, l \mapsto 1, \quad a, b, c, d, f, h, i, k, m \mapsto 0$$

O. Salon, Suites automatiques à multi-indices, *Séminaire de théorie des nombres*, Bordeaux, 1986–1987, exposé 4.

Shape-Symmetric morphism [A. Maes '99]

If P is the infinite bidimensional picture that is the fixpoint of φ, then for all $i, j \in \mathbb{N}$, if $\varphi(P_{i,j})$ is a block of size $k \times \ell$ then $\varphi(P_{j,i})$ is of size $\ell \times k$



$a \mapsto$

a	*b*
c	*d*

$\mapsto$

a	*b*	*i*
c	*d*	**e**
i	**j**	*i*

$\mapsto$

a	*b*	*i*	*i*	*m*
c	*d*	**e**	*h*	*d*
i	**j**	*i*	*f*	*b*
i	*m*	*k*	*i*	*m*
h	*d*	*c*	*h*	*d*

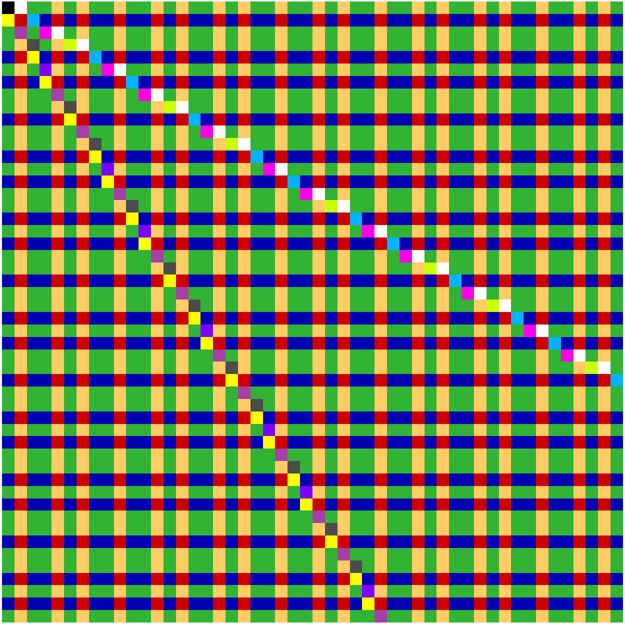
sizes : 1, 2, 3, 5

$\cdots \mapsto$

a	*b*	*i*	*i*	*m*	*i*	*m*	*i*
c	*d*	**e**	*h*	*d*	*h*	*d*	*h*
i	**j**	*i*	*f*	*b*	*i*	*m*	*i*
i	*m*	*k*	*i*	*m*	**g**	*b*	*i*
h	*d*	*c*	*h*	*d*	*h*	*d*	**e**
i	*m*	*i*	**l**	*m*	*i*	*m*	*i*
h	*d*	*h*	*c*	*d*	*h*	*d*	*h*
i	*m*	*i*	*i*	**j**	*i*	*m*	*i*

$\mapsto \cdots$

size : 8,. . .



MORPHISMS $\longrightarrow$ AUTOMATA

We can do the same as for the unidimensional case :
Automaton with input alphabet

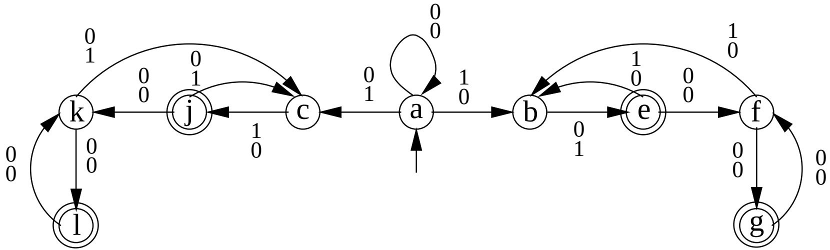
$$\left\{ \binom{0}{0}, \binom{1}{0}, \binom{0}{1}, \binom{1}{1} \right\}$$

$$\varphi(r) = \begin{array}{|c|c|}\hline s & t \\ \hline u & v \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline s & t \\ \hline \end{array}, \quad \begin{array}{|c|}\hline s \\ \hline u \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline s \\ \hline \end{array}$$

we have transitions like

$$r \xrightarrow{\binom{0}{0}} s, \quad r \xrightarrow{\binom{1}{0}} t, \quad r \xrightarrow{\binom{0}{1}} u, \quad r \xrightarrow{\binom{1}{1}} v.$$

We get (after trimming useless part with four states)



This automaton accepts the words

$$\begin{pmatrix} 0w_1 \cdots w_\ell \\ w_1 \cdots w_\ell 0 \end{pmatrix} \text{ and } \begin{pmatrix} w_1 \cdots w_\ell 0 \\ 0w_1 \cdots w_\ell \end{pmatrix}$$

where $w_1 \cdots w_\ell$ is a valid F-representation ending with an even number of zeroes.

Such a characterization is well-known, but differs from the one we get previously...

Reminder

For all $n \geq 1$, we have

$$\begin{aligned} A_n &= \pi_F(\rho_F(n-1)0)+1 \\ B_n &= \pi_F(\rho_F(A_n-1)1)+1. \end{aligned}$$

It is hopefully the same, but **why ?**

- First case : $\rho_F(n-1) = u0$

$$\rho_F(A_n) = \rho_F(\pi_F(\underbrace{\rho_F(n-1)0}_{u00}) + 1) = u01 \text{ no zero}$$

$\rho_F(A_n - 1) = u00$ and

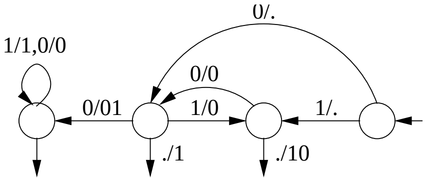
$$\rho_F(B_n) = \rho_F(\pi_F(\underbrace{\rho_F(A_n-1)1}_{u001}) + 1) = u010 \text{ one zero}$$

- Second case : $\rho_F(n-1) = u01$

$$\rho_F(A_n) = \rho_F(\pi_F(\underbrace{\rho_F(n-1)0}_{u010}) + 1) = \text{“}u011\text{”} \ldots$$

Normalize $u011$ or look for the successor of $u010$

Use the transducer (R to L) computing the successor [Frougny'97]



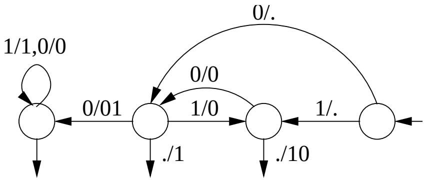
$$10 \rightarrow 100, \quad \text{2 zeroes}$$

$$\underbrace{x10(01)^n}_{u}010 \rightarrow x101(00)^n00 \quad 2n+2 \text{ zeroes}, \; n \geq 0$$

$$\underbrace{1(01)^n}_{u}010 \rightarrow 100(00)^n00 \quad 2n+4 \text{ zeroes}, \; n \geq 0$$

$\rho_F(A_n - 1) = u010$ and

$$\rho_F(B_n) = \rho_F(\pi_F(\underbrace{\rho_F(A_n - 1)1}_{u0101}) + 1) = \text{“}u0102\text{''} \ldots$$



$$101 \rightarrow 1000, \quad \text{3 zeroes}$$

$$\underbrace{x10(01)^n}_{u}0101 \rightarrow x101(00)^n000 \quad 2n + 3 \text{ zeroes}, \; n \geq 0$$

$$\underbrace{1(01)^n}_{u}0101 \rightarrow 100(00)^n000 \quad 2n + 5 \text{ zeroes}, \; n \geq 0$$

Conclusion : “A_n even number of zeroes, B_n one more”, OK

Extension preserving set of $\mathcal{P}$-positions

To decide whether or not a move can be adjoined to Wythoff's game without changing the set K of $\mathcal{P}$- positions, it suffices to check that it does not change the stability property K.

Remark : absorbing property holds true whatever the adjoined move is.

Consequence

A move (i, j) can be added IFF it prevents to move from a $\mathcal{P}$-position to another $\mathcal{P}$-position.

In other words, a necessary and sufficient condition for a move $(i, j)_{i<j}$ to be adjoined is that it does not belong to

$$\{(A_n - A_m, B_n - B_m) : n > m \geq 0\} \cup \{(A_n - B_m, B_n - A_m) : n > m \geq 0\}$$

Thanks to the previous characterizations of A_n, B_m,

Proposition

A move $(i,j)_{i<j}$ can be adjoined to without changing the set of $\mathcal{P}$-positions IFF

$$(i,j) \neq (\lfloor n\tau \rfloor - \lfloor m\tau \rfloor, \lfloor n\tau^2 \rfloor - \lfloor m\tau^2 \rfloor) \; \forall n > m \geq 0$$

and

$$(i,j) \neq (\lfloor n\tau \rfloor - \lfloor m\tau^2 \rfloor, \lfloor n\tau^2 \rfloor - \lfloor m\tau \rfloor) \; \forall n > m \geq 0$$

For all $i, j \geq 0$, $W_{i,j} = 0$ IFF Wythoff's game with the adjoined move (i, j) has Wythoff's sequence as set of $\mathcal{P}$-positions,

$$
(W_{i,j})_{i,j \geq 0} = \left| \begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \\
0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & & & & & & \ddots
\end{array} \right.
$$

Corollary

Let $I \subseteq \mathbb{N}$. Wythoff's game with adjoined moves

$$\{(x_i, y_i) : i \in I, x_i, y_i \in \mathbb{N}\}$$

has the same sequence (A_n, B_n) as set of $\mathcal{P}$-positions

IFF

$W_{x_i,y_i} \neq 1$ for all $i \in I$.

ARE WE DONE ? **Complexity issue**

We investigate tractable extensions of Wythoff's game, we also need to test these conditions in polynomial time. And the winner can consummate a win in at most an exponential number of moves.

MANY "EFFORTS" LEAD TO THIS

For any pair (i, j) of positive integers, we have $W_{i,j} = 1$ if and only if one the three following properties is satisfied :

- $(\rho_F(i-1), \rho_F(j-1)) = (u0, u01)$ for any valid F-representation u in $\{0,1\}^*$.
- $(\rho_F(i-2), \rho_F(j-2)) = (u0, u01)$ for any valid F-representation u in $\{0,1\}^*$.
- $(\rho_F(j - A_i - 2), \rho_F(j - A_i - 2 + i)) = (u1, u'0)$ for any two valid F-representations u and u' in $\{0,1\}^*$.

Morphic characterization of W... in progress

$$\psi : a \mapsto \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad b \mapsto \begin{array}{|c|}\hline e \\ \hline f \\ \hline\end{array} \quad c \mapsto \begin{array}{|c|c|}\hline e & h \\ \hline\end{array} \quad d \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad e \mapsto \begin{array}{|c|c|}\hline j & k \\ \hline l & m \\ \hline\end{array}$$

$$f \mapsto \begin{array}{|c|c|}\hline g & b \\ \hline\end{array} \quad g \mapsto \begin{array}{|c|c|}\hline y & b \\ \hline o & t \\ \hline\end{array} \quad h \mapsto \begin{array}{|c|}\hline z \\ \hline c \\ \hline\end{array} \quad i \mapsto \begin{array}{|c|c|}\hline i & n \\ \hline o & d \\ \hline\end{array}$$

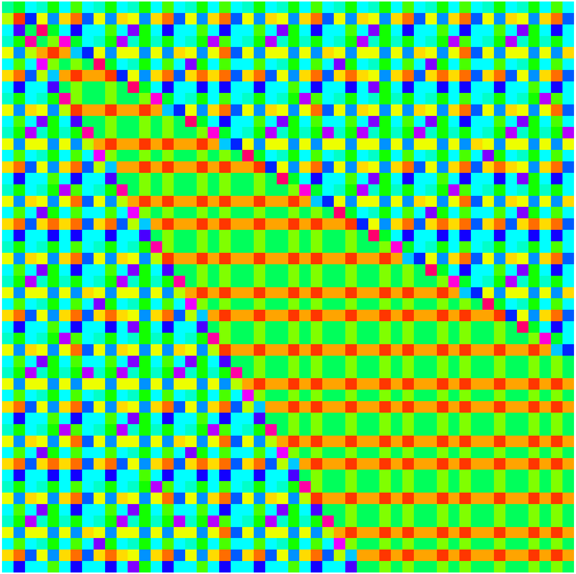
$$j \mapsto \begin{array}{|c|c|}\hline e & p \\ \hline q & r \\ \hline\end{array} \quad k \mapsto \begin{array}{|c|}\hline e \\ \hline s \\ \hline\end{array} \quad l \mapsto \begin{array}{|c|c|}\hline e & u \\ \hline\end{array} \quad m \mapsto \begin{array}{|c|}\hline e \\ \hline\end{array}$$

$$n \mapsto \begin{array}{|c|}\hline i \\ \hline o \\ \hline\end{array} \quad o \mapsto \begin{array}{|c|c|}\hline i & n \\ \hline\end{array} \quad p \mapsto \begin{array}{|c|}\hline e \\ \hline q \\ \hline\end{array} \quad q \mapsto \begin{array}{|c|c|}\hline e & p \\ \hline\end{array} \quad r \mapsto \begin{array}{|c|}\hline e \\ \hline\end{array}$$

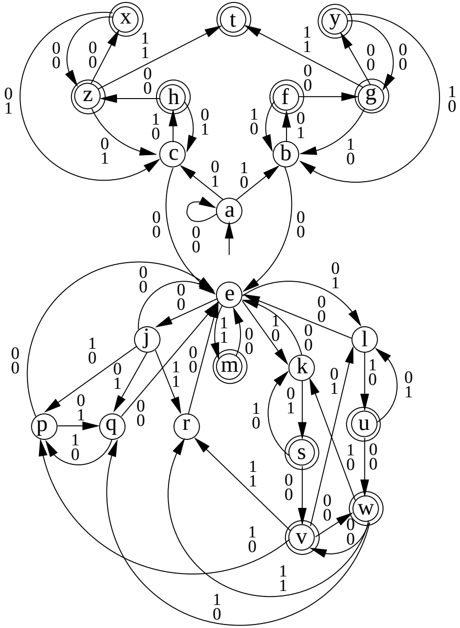
$$s \mapsto \begin{array}{|c|c|}\hline v & k \\ \hline\end{array} \quad t \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad u \mapsto \begin{array}{|c|}\hline w \\ \hline l \\ \hline\end{array} \quad v \mapsto \begin{array}{|c|c|}\hline w & p \\ \hline l & r \\ \hline\end{array} \quad w \mapsto \begin{array}{|c|c|}\hline v & k \\ \hline q & r \\ \hline\end{array}$$

$$x \mapsto \begin{array}{|c|c|}\hline z & n \\ \hline c & d \\ \hline\end{array} \quad y \mapsto \begin{array}{|c|c|}\hline g & b \\ \hline o & d \\ \hline\end{array} \quad z \mapsto \begin{array}{|c|c|}\hline x & n \\ \hline c & t \\ \hline\end{array}$$

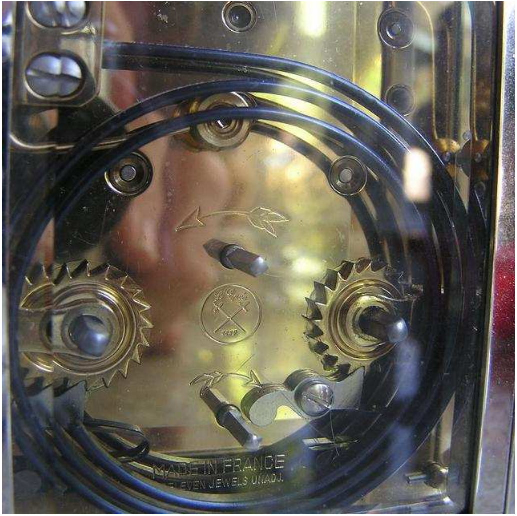
and the coding $\nu : a, b, c, d, e, i, j, k, l, n, o, p, q, r \mapsto 0$ $f, g, h, m, s, t, u, v, w, x, y, z \mapsto 1$.



Corresponding automaton



SOME OF THE MACHINERY BEHIND



Lemma

Let $\mathcal{F}_n$ be the prefix of $\mathcal{F}$ of length n.
For any finite factor *bua* occurring in $\mathcal{F}$ with $|u| = n$,
we have $|u|_a = |\mathcal{F}_n|_a$ and $|u|_b = |\mathcal{F}_n|_b$.

Example

Take $u = aabaab$, *bua* of length 8 starts in $\mathcal{F}$ from position 7.
$\mathcal{F}_6 = abaaba$ is a permutation of u.

$$\mathcal{F} = \underbrace{abaaba}_{\mathcal{F}_6}\overbrace{b\,\underbrace{aabaab}_{u}\,a}^{bua}baababaaba\cdots$$

Proof : algebraic

Lemma

Let $n \geq 1$ be such that $B_{n+1} - B_n = 2$. Then $\rho_F(B_n - 1)$ ends with 101.

Proof : Morphic structure of $\mathcal{F}$

Proposition

$$\{(A_j - A_i, B_j - B_i) \mid j > i \geq 0\} = \{(A_n, B_n) \mid n > 0\}$$
$$\cup \{(A_n + 1, B_n + 1) \mid n > 0\}$$

Proof : Density of the $\{n\,\tau\}$'s in $[0, 1]$

Lemma

Let $u1 \in \{0,1\}^*$ be a valid F-representation. If $\rho_F(\pi_F(u1)+n)1$ is also a valid F-representation, then

$$\pi_F(\rho_F(\pi_F(u1)+n)1) = \pi_F(u00) + \pi_F(\rho_F(n-1)0) + 4.$$

Otherwise, $\rho_F(\pi_F(u1)+n)1$ is not a valid F-representation and

$$\pi_F(\rho_F(\pi_F(u1)+n)0) = \pi_F(u00) + \pi_F(\rho_F(n)0) + 2.$$

Proof : Morphic structure of $\mathcal{F}$

Theorem

Let i, j be such that $A_j - B_i = n > 0$. We have

$$B_j - A_i = B_i + A_n + 1.$$

Concluding result

Theorem

There is no redundant move in Wythoff's game. In particular, if any move is removed, then the set of $\mathcal{P}$-positions changes.

An open problem

- Sprague-Grundy function Mex(Opt(p)) for Nim is 2-regular (*i.e.*, finitely generated 2-kernel)
- so what for Wythoff's game ?

	0	1	2	3	4	5	6	7	8	9	⋯
0	0	1	2	3	4	5	6	7	8	9	⋯
1	1	2	0	4	5	3	7	8	6	10	
2	2	0	1	5	3	4	8	6	7	11	
3	3	4	5	6	2	0	1	9	10	12	
4	4	5	3	2	7	6	9	0	1	8	
5	5	3	4	0	6	8	10	1	2	7	
6	6	7	8	1	9	10	3	4	5	13	
7	7	8	6	9	0	1	4	5	3	14	
8	8	6	7	10	1	1	5	3	4	15	
9	9	10	11	12	8	7	13	14	15	16	
⋮											⋱

A. S. Fraenkel, the Sprague-Grundy function for Wytoff's game, **TCS**'90